THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE FIRST YEARBOOK

A GENERAL SURVEY OF PROGRESS IN THE LAST TWENTY-FIVE YEARS

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THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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The National Council of Teachers of Mathematics has for its objective the advancement of mathematics teaching in junior and senior high schools. All persons interested in mathematics and mathematics teaching are eligible to membership. All members receive the official journal of the National Council—*The Mathematics Teacher*—which appears monthly, except June, July, August, and September. The subscription price is $2.00 per year. Correspondence relating to editorial matters, subscriptions, advertising, and other business matter, should be addressed to Dr. John R. Clark.

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FOREWORD

At the 1925 annual meeting of the National Council of Teachers of Mathematics, a motion was passed directing the President to appoint a committee to prepare and present a Yearbook at the next meeting. This volume represents an effort to fulfill the obligation of that motion. Such merit as it may have obviously belongs to the many who have contributed to it. The President of the National Council and the Chairman of the Yearbook Committee assume responsibility for the mistakes in judgment and for mechanical errors.

After discussion with many teachers a General Survey of Progress in the Past Twenty-five Years in the Teaching of Mathematics in the United States was chosen as the general theme of the Yearbook.

Since this era of progress seems to have been inspired by the Address of Professor E. H. Moore before the American Mathematical Society in 1902, it is fitting that his address should have a prominent place in this book. To Professor Moore and to all the others who have contributed so ably and generously the committee and all teachers of mathematics are under a great obligation.

The members of the Committee and the Council are especially indebted to Professor Raleigh Schorling for editing the publication of this volume. It is due to his faithful work that the book is ready for distribution at the appointed time.

With the hope that this Story of Progress will inspire more interesting and effective work in all the schools of the country, this Yearbook is respectfully submitted.

Yearbook Committee

Charles M. Austin, Chairman.
Harry English
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A GENERAL SURVEY OF THE PROGRESS OF MATHEMATICS IN OUR HIGH SCHOOLS IN THE LAST TWENTY-FIVE YEARS

By David Eugene Smith

I. EARLY ATTEMPTS AT IMPROVING THE SYLLABI

At the beginning of the present century the syllabi in mathematics in the American high schools were determined largely by the requirements for entering our colleges. As a rule examinations were set by each college for its own candidates, the requirements being dictated by the department of mathematics.

As President Butler said in an address delivered on November 6, 1925,—"Twenty-five years ago, the colleges throughout the United States were going their several ways with sublime unconcern for the policies of other colleges, for the needs of the secondary schools, or for the general public interest. They regarded themselves as wholly private institutions and each indulged in some peculiar idiosyncrasy having to do with the admission of students to its freshman class. The colleges made no attempt to agree among themselves, either as to what subjects should be prescribed for admission or what content should be given to any particular subject. The several colleges held admission examinations when it was most convenient for them to do so, and, with the rarest of exceptions, only at the college itself. No secondary school could adjust its work and its program to the requirements of several colleges without a sort of competence as a pedagogic acrobat that was rare to the point of non-existence. The situation would have been comic were it not so preposterous."

The purpose of the examination at that time seems to have

been chiefly to assure the entrance of students who gave at least a fair degree of promise of becoming mathematicians. Although set by such a large number of different examining bodies, the subject matter was fairly uniform, based as it was upon a tradition that was generally known throughout the country.

In 1902 a committee of the American Mathematical Society, on "definitions of college entrance requirements," made a recommendation that elementary algebra should cover the usual topics through progressions; that higher algebra should cover permutations and combinations, the applications of mathematical induction, logarithms, theory of equations (with graphic methods), Horner's method, determinants, and complex numbers; that plane geometry should cover "the usual theorems and constructions of good standard text-books, the solution of original exercises, applications to problems of mensuration of lines and of plane figures, and to loci problems," and similarly for solid geometry. Plane trigonometry was to cover the six functions, the "proofs of principal formulas," logarithms, and the solution of triangles.

This report was evidently rather inclusive through its very lack of precision. It kept open the way for every eccentric examiner to propose almost any question he wished, and yet it served fairly well as a starting point for reform. At any rate, it was the expression of a national instead of a local opinion.

In the year 1900 the College Examination Board was organized. This was a great step in advance. It sought to unify the examinations and to prepare them with much greater care than was usually the case with local efforts. It also gave an opportunity for the schools to be consulted by and become a part of a central organization, thus being represented in the preparation of the papers. While the range of the examinations soon became that which was set by the committee of the American Mathematical Society, and was therefore rather indefinite as to limitations, the papers themselves became more standardized and represented in general a

better selection of material. The traditional still played a leading role, but at last there was some hope of modernizing the syllabus and there was a feeling of assurance that this improvement would in due time be realized.

Tradition still demanded the retention of a large amount of abstract manipulation of polynomials, including long problems in the multiplication and division of integral and fractional expressions, with extended work in the finding of roots, in factoring, in lowest common multiple, and in highest common factor, and with equally useless manipulations of complex fractions and radicals. Simultaneous linear equations extended to four and more unknowns, and simultaneous quadratics of the trick variety were in evidence. As to the higher algebra it is not necessary to speak, since that concerned and still concerns a relatively small number of pupils, these being a rather selected lot who always look forward to a certain amount of work in college mathematics.

Geometry was still more stagnant, as would naturally be expected of a subject that had been many centuries longer in the making.

In neither subject did there seem to be any clear conception of the purpose of teaching mathematics in the twentieth century as distinguished from that which came into being with the rise of analysis and algebraic symbolism three hundred years ago.

It is hardly necessary to consider other syllabi. In general they were more conservative than the somewhat indefinite ones followed by the College Entrance Examination Board and suggested by the committee of the American Mathematical Society, and they showed a lower range of scholarship.

II. INFLUENCES FOR THE BETTERMENT OF THE COURSES

In the period in question there were various influences making for the betterment of the syllabi and of the courses offered in the schools. These include the following:

1. The work of the International Commission on the Teach-

ing of Mathematics, serving as it did, to let American teachers see the curricula of the best types of secondary schools in all the other leading countries of the world and to compare our progress with that which is found abroad. Naturally this comparison showed us that conservatism existed elsewhere as well as here, but it also served to show that various other countries were ahead of us in achievement and that it was well for us to ascertain the cause and to see if we had any advantages to counterbalance this apparent disadvantage. A brief statement of the work of the Commission is set forth in Section III of this survey.

2. The work of the National Committee on Mathematical Requirements. This committee was appointed by the Mathematical Association of America in 1916 and was financially assisted by the General Education Board. Its investigations were thorough and its report was fully considered by representative associations of teachers throughout the country. It made a careful study of the purposes which should determine the teaching of mathematics in our secondary schools and suggested a syllabus which should eliminate non-essentials, retain those things which should best meet the needs of pupils of the present generation, and introduce such modern material as should strengthen the work without attempting to make it unreasonably difficult. The nature of the report of this committee is briefly summarized in Section IV. Since certain portions of the report itself were the object of study and discussion all over the country for at least two years before they were finally approved, and since the complete report has been very widely circulated and discussed, its influence has been very great.

3. The revised requirements of the College Entrance Examination Board. These were prepared by a commission appointed by the Board in 1922. They were reported in 1923 and are now the basis of the examinations. They represent the combined judgment of the colleges and the secondary schools and are thus no longer subject to the criticism that the college is assuming undue rights in dictating what shall be taught

in the high school. They are more precise than any others that have been heretofore set forth for the guidance of schools, and even where schools are not preparing for these examinations they have tended to set a new standard and to eliminate much of the work for which no reasonable justification could be found. The work of this commission is set forth more fully in Section V.

4. The rise of the Junior High Schools. The rise of these schools, a development of the "six-and-six plan," is one of the most significant movements in the last quarter of a century. Not hampered in its early development by any examination system, this type of school was free to formulate its own course and to set its own standards. The result was a more uniform curriculum than at first was thought probable. It enabled our schools to introduce intuitive geometry in a satisfactory manner in Grade VII, to allow algebra to grow naturally out of the need for the formula in connection with this geometry, to show that a simple form of trigonometry grows out of algebra, and to give some slight notion of the significance of a demonstration. At the same time it continued the work in arithmetic as applied to the intuitive treatment of geometry, as related to algebra, and as bearing upon the every-day needs of our people. The result was a perfectly natural correlation of the various parts of elementary mathematics that are suitable for this school period, a process of discovery of mathematical ability, and an interesting type of work that had been lacking in the older kind of arithmetic and algebra which it displaced. Courses differed in different schools, but this was rather in unimportant details than in the large features. Although the plan has not met with universal approval, the general feeling has been very favorable to it and, with progressive and well-trained teachers, the new curriculum has been decidedly successful.

5. The work of the schools of education in the universities. As effective agencies in any noteworthy improvement in teaching, these schools may be said to be largely a product of the twentieth century. Naturally their achievements have been

manifested largely in the line of mental measurements which, for our purposes include various types of tests. The results have been very encouraging, although the published tests have, as stated later, had one unfortunate feature that has to a great extent counterbalanced the good which might have been accomplished and which will eventually result when they are prepared with more care. In general, however, these schools of education have shown that mathematics can easily be adjusted to the capacities of young people, while the capacities of these pupils cannot be so readily adjusted to the old-style mathematics; that the science can be made part of the lives of children as well as those of adults; and that students may rightly expect to enjoy learning as they enjoy other phases of life. Whether this attitude of mind in the work of the school has lowered our ideals of scholarship is a mooted question, but that it must necessarily do so can hardly be asserted by anyone who carefully considers the future work of our schools. In any case our departments of education have, through their experiments and their studies, created a healthy spirit in our schools and have of late become more internationally minded in their outlook. Although they have not fostered sound mathematical scholarship as much as we might wish, they have probably done so as fully as present conditions permit, and it must be recalled that their work is still in its infancy.

6. The textbook. Among the various influences that have worked for the progress of mathematics in the last twenty-five years it is but a just tribute to the makers of textbooks to say that the progress that has been made would have been absolutely impossible without their aid. While certain books have appeared that were impossible as aids to instruction, and while commercialism occasionally enters into an effort of this kind, it is only fair to say that a large majority of textbooks on mathematics have been prepared solely with a view to assist in the bettering of instruction in our elementary and high schools. There is no other country in the world that produces as fine pieces of bookmaking as those issued from our best presses. At the same time, in no other country is the work

set forth in such textbooks with as great attention to the needs and interests of children as is found here, and in none has the advance in teaching been so rapid. On the other hand, in sound scholarship many foreign books surpass ours, but not in meeting the practical needs of the people.

7. The Spirit of the Times,—a phrase which certain writers assert has no meaning, but which is convenient as representing the mass psychology of the moment. At any rate, within the past quarter of a century there has been a general recognition the world over that the traditional education of the nineteenth century is not adapted to present conditions; that there must be a well-accepted reason for teaching algebra or else the subject must be discarded, and similarly for the other mathematical disciplines. The result of this feeling has been very salutary as may be seen in the present requirements in mathematics as set by the College Entrance Examination Board.

III. THE WORK OF THE INTERNATIONAL COMMISSION

The section on philosophy, history, and instruction of the Fourth International Congress of Mathematicians, held in Rome, April 6 to 11, 1908, submitted to the Congress a resolution to create an International Commission on the Teaching of Mathematics. The suggestion was indorsed by the Congress on April 11, and an organizing committee was appointed consisting of Professor Klein of Göttingen, Sir George Greenhill of London, and Professor Fehr of Geneva, besides others appointed immediately thereafter. Delegates were afterwards chosen from the countries which had taken part in at least two of the congresses. The result was the publication of a large number of reports showing the nature of the work done in mathematics in schools of all types throughout most of the world. In the United States reports were prepared relating to various topics, including the following:

(1) Mathematics in the elementary schools; (2) Mathematics in the public and private secondary schools; (3) Training of teachers of elementary and secondary mathematics; (4)

Influences tending to improve the work of the teacher of mathematics; (5) Mathematics in the technological schools of collegiate grade; (6) Undergraduate work in mathematics in colleges of liberal arts and universities; (7) Mathematics at West Point and Annapolis; (8) Graduate work in mathematics in universities; (9) Report of the American committee; (10) Curricula in mathematics in various countries; (11) Mathematics in the lower and middle commercial and industrial schools of various countries; (12) The training of elementary school teachers in mathematics in various countries; 13) The training of teachers of mathematics for the secondary schools in various countries. Allied to this work there were published two other bulletins as follows: (1) Bibliography of the teaching of mathematics, 1900-1912; (2) Union list of mathematical periodicals taken by the larger libraries in the United States.

These reports were published by the United States Bureau of Education in the years 1911-1918 and were widely circulated in this country. They served to show to our schools the range of our system on instruction in mathematics and the general purposes in view in the various types of school. Perhaps the chief value to our country, however, was the comparison which was thus made possible between the work done here and that done in the other leading countries of the world. This showed that we were distinctly behind other countries, as to subject matter, particularly after Grade IV, although we might properly claim to be at least equal to them in the spirit of the work done in our schools. It raised the question, however, as to whether a good spirit could compensate for poor work, and it caused a large amount of discussion in bodies of teachers throughout the county. The past ten years have shown some gratifying results of this discussion.

IV. THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS

The work of the National Committee is too well known for detailed remarks. It is set forth in its report, *The Reorganiza-*

tion of Mathematics in Secondary Education, published in 1923 by the Mathematical Association of America, under whose auspices the Committee was established. This report was prepared in close cooperation with bodies of teachers throughout the country. It set forth very clearly the aims of mathematical instruction in the several years of the junior high school, the senior high school, and the older type of four-year high school. It presented the model courses for these several types of school and made suggestions for carrying out the work. It considered the question of college entrance requirements, the basal propositions of geometry, the role of the function concept, and the terms and symbols which might properly have place in the schools. It fostered various other investigations, including the present status of the theory of disciplinary values, the theory of correlation applied to school grades, a comparison of our curricula with those in use abroad, experimental courses in mathematics, standardized tests, and the training of teachers.

It is not too much to say that the advance in the teaching of mathematics in our secondary schools in the last decade has been due in large part to the work of this committee. Since the report is available in most high school and public school libraries in the country, it would only seem to lessen its value to attempt any further résumé of its contents.

V. THE INFLUENCE OF THE COLLEGE ENTRANCE EXAMINATION BOARD

No other influence in the reform of the teaching of mathematics in our secondary schools of the present time has been more potent than that exerted by this board. Often thought to be very conservative and to represent the views of the college professor alone, it has shown itself in the best sense radical in its reforms, and representative of the secondary schools to fully as great a degree as of the colleges. Among its reforms only the most noteworthy can be mentioned in this brief report.

In algebra it has eliminated the extended and largely use-

less manipulation of polynomials in connection with the elementary operations. It distinctly says:

"It is not expected that pupils will be called upon to perform long and elaborate multiplications or division of polynomials, but that they will have complete mastery of those types that are essential in the subsequent work with ordinary fractional equations, and with such other topics as are found in elementary algebra. In other words, these operations should be looked upon chiefly as a means to an end."

Thus in two sentences it has struck out a large amount of entirely useless and uninteresting work that had cumbered up the inherited course.

It then eliminated most of the work in factoring, a subject which began to occupy an undue amount of space in the closing quarter of the nineteenth century, reaching its culmination at the opening of the present one. The requirement was reduced to only three types,—

(1) Monomial factors;
(2) The difference of two squares;
(3) Trinomials of the type x^2+px+q.

When we consider the fact that the subject was used almost exclusively in those fractions and equations which were made up merely for the purpose of using it, the board's attitude was most salutary. Indeed, from the standpoint of practical use it may be doubted if the value of the factoring of a quadratic trinomial is not even now overrated.

The requirement in fractions has been simplified, and the result has been the elimination of long and useless operations that consumed time and led to no worthy end. The report says:

"The meaning of the operations with fractions should be made clear by numerical illustrations, and the results of algebraic calculations should be frequently checked by numerical substitution as a means of the attainment of accuracy in arithmetical work with fractions."

The requirement includes complex fractions of about the following degree of difficulty:

$$\frac{p+\frac{a}{b}}{q-\frac{c}{d}}, \qquad \frac{\frac{a+3b}{c-5d}}{\frac{a-3b}{c+5d}}, \qquad \frac{\frac{a}{b}+\frac{c}{d}}{\frac{m}{n}-\frac{p}{q}}.$$,,

This is a great gain over the plan of even a dozen years preceding. Such forms as those here given actually enter into the simple formulas which the pupil will see in elementary science. They can therefore be justified.

Furthermore, ratio is treated merely as a case of simple fractions, and proportion is treated as a simple type of fractional equation, so that at once the whole subject has been simplified materially,—indeed, as a separate topic, it is substantially discarded. Such terms as "alternation" and "composition" have naturally been abandoned by this action.

The position of the formula is a great advance over what was the case a generation ago. Few would now deny that the formula is the element of algebra that will be most often used by the student when he begins his serious work in science. This should therefore be the subject of greatest emphasis in the first year's work. The commission of the Board recognizes the fact and has this to say with respect to promoting the subject:

"In the work done with formulas, the general idea of the depedence of one variable upon another should be repeatedly emphasized. The illustrations should include formulas from science, mensuration, and the affairs of everyday life. Throughout the course, there should be opportunity for a reasonable amount of numerical work and for the clarification of arithmetical processes."

The graph also has the proper kind of recognition, being introduced for the definite purpose of illustrating and making more clear the formulas needed in science and in business.

In the subject of linear equations the pupil is no longer expected to solve an array of abstract types that admit of no application, but to devote his energies to the solution of those

types which have some chance of being used. For this reason the first year's work excludes cases with three or more unknowns. The report further makes the statement:

"Besides numerical linear equations in one unknown, involving numerical or algebraic fractions, the pupil will be expected to solve such literal equations as contribute to an understanding of the elementary theory of algebra. For example, he should be able to solve the equation

$$s=\frac{ar^n-a}{r-1} \qquad \text{for } a.$$

"In the case of simultaneous linear equations, he should be able to solve such a set of equations as

$$\begin{aligned} ax+by&=k,\\ cx+dy&=l, \end{aligned}$$

in order to establish general formulas. But the instruction should include a somewhat wider range of cases, as for example:

$$\begin{cases} ax+(a+b)y=ab,\\ ax+(a-b)y=-ab, \end{cases} \qquad \text{or} \qquad \begin{cases} ax+by=ab,\\ x+\left(1+\frac{b}{a}\right)y=a. \end{cases}$$

"The work in equations will include cases of fractional equations of reasonable difficulty; but, in general, cases will be excluded in which long and unusual denominators appear and in which the common factor of the denominators, or the lowest common denominator, cannot be found by inspection.

"Problems in linear equations, as in ratio, proportion, and variation, will whenever practicable be so framed as to express conditions that the pupil will meet in his later studies."

In the latter part of the first year's work two notable improvements have been made. The first relates to the simplification of the work in surds, an inheritance from the past that has lost much of its former significance. The report considers exponents and radicals under the following head:

1. "The proof of the laws for positive integral exponents.

2. "The reduction of radicals, confined to transformation of the following types:

$$\sqrt{a^2b}=a\sqrt{b}, \qquad \sqrt{\frac{a}{b}}=\frac{\sqrt{ab}}{b} \qquad \frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$$

$$\frac{1}{\sqrt[3]{a}}=\frac{\sqrt[3]{a^2}}{a}, \qquad \frac{1}{\sqrt[3]{a^2}}=\frac{\sqrt[3]{a}}{a},$$

and to the evaluation of simple expressions involving the radical sign.

3. "The meaning and use of fractional exponents, limited to the treatment of the radicals that occur under 2) above.

4. "A process for finding the square root of a number, but no process for finding the square root of a polynomial.

"In all work involving radicals, such theorems as

$\sqrt{ab}=\sqrt{a}\sqrt{b}$ and $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$ may be assumed. Proofs of these theorems should be given only in so far as they make clearer the reasonableness of the theorems; and the reproduction of such proofs is explicitly excepted from the requirements here formulated."

The second noteworthy feature of the latter part of the first year is the introduction of simple numerical trigonometry. This is made possible by the elimination of a considerable amount of relatively useless material and by the selection of the minimum essentials of the subject.

Upon the range of this work the report recommends the following:

"The use of the sine, cosine, and tangent in solving right triangles.

"The use of four-place tables of natural trigonometric functions is assumed, but the teacher may find it useful to include some preliminary work with three-place tables.

"The recognition of the fact that the pupil should acquire facility in simple interpolation; in general, emphasis should be laid on carrying the computation to the limit of accuracy permitted by the table."

With respect to geometry the Commission makes three noteworthy recommendations:

1. That the number of "book theorems" required on any examination shall be materially reduced; in fact, only eighty-nine theorems are included in the syllabus for plane geometry, and of these only about a third are required for examination purposes. This allows plenty of time for the important subject of "originals," a subject which has assumed an entirely new position of importance within the last quarter of a century.

2. That a year's course involving both plane and solid geometry be allowed in place of the single course in plane geometry. This allows a pupil to secure a fair knowledge of both phases of the subject in a single year. This is rendered possible by the reduction to fifty-nine of the number of propositions in plane geometry and to twenty-four as the number required for examination, with a similar reduction in solid geometry.

3. That there be offered a certain amount of work in mensuration of a type more frequently met with in various lines of industry. The treatment of this work is modern in spirit and the work itself is outlined in the commission's report.

In brief, the report shows a tendency to break away from too much formalism, to depend much more upon originals, to combine plane and solid geometry (if desired) in a single year, and to approach European standards in the field of practical mensuration.

VII. THE PROGRESS OF ARITHMETIC

Some idea of the progress of arithmetic, of the type used in the junior high school, in the period in question may be obtained by a comparison of the nature of the topics as set forth in some of the most prominent arithmetics of the close of the

nineteenth century with that of the present time. The following synoptic presentation of the case shows the nature of the changes that have taken place in this brief period:*

Then	*Now*
Arithmetic of special and unusual occupations. For example,	Arithmetic of the daily life of the people. For instance,
alligation	arithmetic of the home
equation of payments	a simple bank account
arbitrated exchange	the check book
partnership involving time	arithmetic of the store
true discount	organization of common corporations
general average	cost of production and overhead charges
tax collectors' commissions	transportation
marine insurance	the common industries
partial payments	farm problems of today
measurement of hogsheads, granaries, and cisterns	
Obsolete processes. For example,	Short methods. For example,
greatest common divisor and least common multiple of large numbers	in making change
	in checking bills
	in common multiplication
Work with long and unusual fractions	Fractions limited to those of everyday life
Arithmetic progression	Thrift and savings
Geometric progression	Safe types of investment
Cube root	Percentage related more closely to decimals
Present worth:	
at simple interest	Decimals related more closely to U. S. money
at compound interest	
Troy weight	Graphs
Extensive work in compound numbers of unusual types	Compound numbers limited to a few really useful types of work
Gregorian and Julian calendars	Our duty to the government
Proportion as a means of solving commercial problems	Government expenses:
	city, state, national:
Ratio without applications	necessity for thrift
No reviews except by going over the same work	Systematic reviews of principles but with new problems
All topics of equal importance	Minimum essentials emphasized

Not only have such changes as these in the topics of arithmetic been made, but even more noteworthy ones appear in the nature of the problems. The following list, taken from a popular textbook of a quarter of a century ago contains a fair

*This synopsis and certain other portions of this section have been taken from the author's essay on *The Progress of Arithmetic in the Last Quarter of a Century*, Boston, 1923.

sampling of what can be found in most of the works of that period:

1. Find the value of $[84-7\times6+(3\times5)-31\div9$.

2. Divide

19/42 of 28/33 of 11/14 of 7 1/9 by 23/35 of 5/8 of 16/23 of 8/35 of 24 5/12.

3. I bought 26 yards of carpet at $1 9/10 a yard, 3 curtains at $5 3/5 each, and 6 chairs at $1¾ each. What is my bill? (As if we ever used these common fractions of a dollar in this way!)

4. A vessel sailed from Portland, Me., for New Orleans with a cargo of 1528,375 tons of ice. On the way 94.58 tons of it melted. How much ice reached New Orleans? (The weight of the cargo of ice is given to within 2lb., which is rather close when we consider that it probably varied 1000 lb. while being stowed away.)

5. Reduce to ounces 5T. 10cwt. 24lb. 8oz.

6. From 5 lb. 7 oz. take 3 lb. 10 oz. 5 drams, 1 scruple, 15 grains.

7. Find the compound interest on $4921.50 for 4 yr. 9 mo. 24 da. at 7%, using the table.

8. What is the present worth of $3180.50 payable in 2 yr. 3 mo. 21 da., when money is worth 5 1/2%?

9. A, B, and C formed a partnership. A put in $3000 for 5 mo., and then increased it $1500 for 4 mo. more. B put in $9000 for 4 mo., and then, withdrawing half his capital continued the remainder 3 mo. longer. C put in $5500 for 7 mo. They gained $3630. What was each partner's share of the gain?

10. If 5 horses eat as much as 6 cattle, and 8 horses and 12 cattle eat 12 tons of hay in 40 da., how much hay will be needed to keep 7 horses and 15 cattle 65 da?

11. Three men bought a grindstone 20 inches in diameter. How much of the diameter must each grind off so as to share the stone equally, making no allowance for the eye?

12. A man bequeathed his property in such a way that his wife received \$7 for every \$5 received by each of his two sons and every \$4 received by each of his three daughters. If his estate was worth \$250,000, what was the sum bequeathed to each of the heirs?

13. Find the greatest common divisor of 462, 882, and 546.

14. A farmer wishes to put 336 bushels of wheat and 576 bushels of corn into the least number of bins possible of uniform size, without mixing the two kinds of grain. How many bushels must each bin hold?

15. Find the least common multiple of 2520 and 2772 and also of 11 1/9, 14 2/7, and 33 1/3.

16. Change 268*te* on the duodecimal scale to the decimal scale.

17. Multiply 3424 on the quinary scale by 234 on the same scale.

18. Take 3/5 of 4 mi. from 7/8 of 3 mi. 18 rd. 3 yd. 2 ft.

19. Divide 19 T. 17 cwt. 29 lb. 7 oz. by 4/5.

20. Find the weight of an ivory ball 2 in. in diameter, the weight of ivory being 1825 oz. a cubic foot. (It would be interesting to see ivory sold by the cubic foot.)

21. A man walked 23 2/3 mi. the first day of a trip, 25 3/20 mi. the second, 28 14/64 mi. the third, and 26 53/100 mi. the fourth. How far did he walk in all?

22. Find the value of $8\ 3/7 + 5\ 4/9 + 9\ 2/3 - 3\ 8/21 - 3\ 6/7$.

23. Find the value of $23/49 \times 7\ 3/4 \times 9/10$.

24. Find the value of $\dfrac{9/19 \text{ of } 13\ 7/12}{18/38 \text{ of } 7\ 5/16}$

25. Reduce 6 mi. 37 rd. 4 yd. 3 ft. 6 in. to inches, and 5/7 of a rod to yards, feet and inches.

26. Reduce 721327 inches to miles. (The number was not even written in periods of three figures.)

27. Reduce 7 sq. mi. 17 A. 13 sq. ch. to square chains.

28. Reduce 9230 scruples to higher denominations.

29. Reduce 7 hr. 32 min. 49 sec. to seconds.

30. Find the sum of 10 mi. 172 rd. 2 yd. 2 ft. 9 in., 12 mi. 172 rd. 4 yd. 11 in., 16 mi. 74 rd. 1 yd. 2 ft. 3 in., 19 mi. 198 rd. 4 yd. 9 in., and 39 mi. 131 rd. 5 yd. 1 ft. 7 in.

Whatever may be said of many of the problems set in our schools today, a reading of the above list shows that there has been a decided advance in the quality of the material and in adapting the exercises to the needs and interests of the pupils.

As to the methods of presenting the subject of arithmetic or, indeed, of the other branches of mathematics, this report is not directly concerned. It is desirable, however, to call attention to one change that has become more and more evident in the last two or three decades, and that is our sympathy with childhood,—not our affection, probably not always our good judgment, but certainly our sympathy with the child in school. The severe discipline of two generations ago had begun to relax at the close of the nineteenth century, and at the present time it has become very much less pronounced in the better type of school. The pupil has come to live the child life more freely instead of trying to live the adult life that the world not long ago sought to impose upon him. We have still a long way to go to reach the goal, and we run continued risk of so reducing the mental food supply as to make education a poor affair and one that requires so little effort as to have neither interest nor value. On the whole, however, the average elementary pupil gets much more joy out of his school life than his parents did out of theirs, and his general range of knowledge is rather better than theirs was at the same age. It is also probable that his powers of computation in those ordinary problems of life that he is capable of understanding do not suffer by the same comparison. In spite of the easy fashion to deny this assertion, there seems no reason to think that it is not perfectly true.

All this has been a distinct gain. It has not come from any of the "methods" that loomed up so large in the eyes of the

teachers of a generation ago, nor has it come to any great extent from the results of psychological studies; it has come largely from the use of plain common sense in adapting arithmetic to this new view of education,—that of letting the child live as natural a life as possible while in school,—and in adapting the work in arithmetic to his mental powers at each stage of his growth.

Perhaps the most important change of all is seen in the purpose of teaching arithmetic. A quarter of a century ago it was felt that the subject should be hard in order to be valuable, and it sometimes looked as if it did not make so much difference to the school as to what a pupil studied so long as he hated it. The old idea that this was good for the mind and soul was not at that time fully discarded. There was also prevalent the idea that as many applications of arithmetic should be introduced as the time allowed, irrespective of whether they were within the mental horizon of the pupil or within the probable needs of his life after leaving school. This view has now been changed; the purpose of teaching arithmetic has come to be recognized as the acquisition of power to calculate within the limits of the needs of the average well-informed citizen. It has also come to be recognized that the problem is primarily designed to show a need for computation, by giving applications that add to the interest in calculation and by introducing the puzzle element of problem-solving, which may add further interest. A secondary purpose of the problem is the imparting of some knowledge of the economic conditions, that the pupil will find in daily life, this being presented to him in a simple manner that will make it seem interesting and worth while.

We should not fail, moreover, to recognize the value of the tests in arithmetic which have been devised during the past quarter of a century. These precede by some years the tests in algebra and geometry and have been much less open to legitimate criticism. They have accomplished much in the improvement of the work in arithmetic, in diagnosing pupils' difficulties, and in the measurement of their capacities.

It is reasonably certain that the newer tests in the high school subjects will, when purged of certain objectionable features, especially as to their work in traditional subject matter, accomplish similarly beneficial results.

VII. THE PROGRESS OF ALGEBRA

Encouraging as has been the progress of arithmetic in our schools, the progress of algebra has been none the less noteworthy. Twenty-five years ago the subject was usually taught as if it were a purely mathematical discipline, unrelated to life except as life might enjoy the meaningless puzzle. Valuable as the teacher might feel it to be, the majority of pupils looked upon it as a fairly interesting way of getting nowhere.

If we were to seek the most significant step taken in the improvement of the teaching of algebra in the last twenty-five years, it would probably be found in the clearer vision that we have of the real purpose of the subject. To take our current educational phraseology, we have been concerned, and properly so, with establishing our "objectives." The purpose a quarter of a century ago seems to have been to make mathematicians; the purpose today is to make well-informed American citizens. A man or a woman is not well informed if he or she is ignorant of the general meaning of geography, of the simpler natural sciences, of a few masterpieces of our language, of the significance of foreign tongues, of the qualities of good art (including music), of the social and economic needs of people, of the nature of government, of one's duties as a citizen, and of the significance of religion,—most of these being taught to best advantage in school. In the same way both the man and the woman needs to know something of the significance of mathematics.

As a result of this view of the reason for teaching algebra, we have come to see that we should not expect everyone to solve two simultaneous quadratic equations, although out of an entire class there will be found a few who can do so. Nor should we expect to have all the pupils able to factor ax^2+bx+c (a useless accomplishment for most people), even though

a considerable number will take pleasure in performing such a task and will thereby acquire some special skill which they may find useful in later work. The purpose of teaching algebra is found in none of these details: it consists in giving to everyone a general idea of the meaning of algebra, together with a few definite and useful applications which everyone is likely to meet. If the subject is to be valuable, the learning should be a pleasure, and it may properly be expected that this pleasure will carry the pupil into such manipulations of algebraic expressions as will fix the habit of using algebra in the cases to which it can be applied.*

This has led to a consideration of those topics of algebra that are of most worth to the average citizen, and herein the change has been very marked. If, today, the concensus of opinion were to be taken among progressive teachers it would probably result in the naming of the formula, the graph, the directed number, the linear equation with one unknown, and (by way of application) numerical trigonometry as the five important topics to be considered. Facility in algebraic manipulation, which played such an unduly important part a generation ago would be relegated to a relatively minor role at the present time. Painfully precise definitions and attempts at ultra scientific explanations are no longer felt to be either necessary or desirable.

One of the most popular texts of twenty-five years ago had eight pages of definitions and theory before a single example was given, and out of nearly 1600 exercises in the first 147 pages only 111 were verbal problems and only two could lay claim to relating to any apparent human need. Another text of that period gave about 1800 exercises in the first 128 pages; of these only 109 were of the verbal variety, and only one had any apparent application to any condition that would arise in daily life.

Any good modern text, however, would show the need for algebra on the first page; would begin its real problems im-

*For an amplification of this subject see the author's *The Progress of Algebra in the Last Quarter of a Century*, Boston, 1925.

mediately; and would give a large number of verbal and written exercises at once, with as many genuine applications as reasonably possible,—applications of a kind that pupils can understand and in which they will have a real interest. Some of our modern books have more verbal problems than both of the other two already referred to, and many times the number of genuine applications relating to daily life.

In neither of these two algebras of a quarter of a century ago (and they were among the best of their time) was there a single example showing the meaning of or the need for the directed number, whereas in a good modern text the pupil will find dozens of them, not to mention numerous illustrations, showing its value in our daily lives.

These are only a few of the evidences of progress in the purpose of school algebra in the last quarter of a century,—a progress which, without exaggeration, may be characterized as revolutionary. Probably no other subject found in the course of study in the average high school has undergone so marked a change.

The earlier type of algebra was arranged on the same plan as the earlier type of arithmetic. On the theory that we must scientifically define all terms before they can safely be used, the book began with definitions—a plan which would make it necessary to define "elephant" before visiting a menagerie. If the book gave any idea at first about the purpose of algebra, it was that it was a science in which letters were used in solving the most impractical sort of number puzzles.

The book next proceeded to introduce strange terms, such as monomial, binomial, residual (now discarded), polynomial, coefficient, and exponent,—not as they were needed, but in order to provide for their use at some time in the future.

It then took up the four operations which had been developed in arithmetic, but which have only a slight use in practical algebra, and spent some weeks of the pupil's time in mastering a technique that was of little value—at least in the beginning of the science.

Having covered this ground for integral expressions, the

book then considered the question of fractions by giving work of a type that few students would ever need in subsequent mathematics. Linear equations were then introduced; after which followed a large amount of work in incommensurable numbers (involving such names as "surds" and "radicals"), and then quadratics, proportions, series, and other advanced topics. There were but few attempts to frame verbal problems, even of the fictitious type, and none to develop the real applications of the science.

At the present time every leading writer of school algebras is making the attempt, with more or less success, to arrange the topics on a more rational basis. The sanely progressive books begin with the formula and show its meaning, its practical use, and the method of deriving one formula from another. This being done, the most valuable part of pure algebra has been presented, and it is a part that, a quarter of a century ago, was practically ignored. The graph, the negative number, and the linear equation are then presented, the equation having already been encountered in connection with the formula. Numerical trigonometry appears later in the course. As to the division of polynomials by polynomials, elaborate algebraic fractions, highest common factor and the lowest common multiple, most of factoring, roots, most of the work in surds, linear equations with more than two unkowns, and simultaneous quadratics,—the relative value of all these has diminished greatly in the estimation of those who wish to salvage the parts of algebra that the pupil will really use in his later work.

In the matter of algebraic problems there has also been a notable advance. Some idea of the types in current use a quarter of a century ago can be formed by considering a few of the best of the problems contained in a popular work of that period:

A man, being asked if he had 100 head of cattle, replied that if he had twice as many as he then had and 4 more, he would have 100. How many had he?

If B were 5 yr. younger, A's age would be twice B's. The

sum of their ages is 20. How old is each?

A's capital was ¾ of B's. If A's had been $500 less, it would have been ½ of B's. What is the capital of each?

Paving a square court with stones at 40c a square yard will cost as much as inclosing it with a fence at $1 yer yard. What is the length of a side of the court?

Bought 8 horses, a number of cows, and 100 sheep for $2500. The number of cows was equal numerically to 4 times the price of a sheep, and a sheep and a horse cost $5 less than 1/5 the cost of all the cows. Find the cost of a horse, and a sheep, and the number of cows, if a cow cost $40. (As an example of English, in which a *number* is equal *numerically,* this is interesting.)

It must not be thought that such problems are without value, that there are no good reasons for giving them, or that the older books are to be condemned for having a reasonable number of this type. Some of them have stood the test of time and have maintained their own throughout the centuries because pupils could easily visualize them and were interested in their solution, which is rarely the case with any of the multitude of real problems of a technical nature in physics, in shop practice, in the biological sciences, or in the field of commerce. It is probable that we shall always find it best to draw upon certain types of puzzle problems as exercises in algebra, in arithmetic, and in various other branches, for the reason that most technical problems are too difficult to be understood by the pupil when he is studying these subjects. To postpone algebra until such time as he could understand these applications would be to put off taking up something for which the pupil is mentally ready until a time when he would deem it too childish to be of interest. Indeed, it may safely be said that we are probably not making enough use of the interest afforded by the puzzle element in any of our work in mathematics.

All this, however, is no excuse for giving nothing but unreal problems in algebra, which was the situation at the beginning of the century. That we have made a gratifying advance is

seen from an examination of various leading textbooks of the present day, the genuine applications of algebra, particularly in the case of formulas and of other types of equations (as in the study of ratio), being much in evidence.

It is, however, in the introduction of numerical trigonometry as a legitimate, interesting, simple, and valuable part of algebra that the most notable step in the last quarter of a century has been taken. It has long been recognized that trigonometry has much more practical importance in the world than most of the work given in the older type of algebras. The tradition that this subject must necessarily follow demonstrative geometry has no merit except its antiquity: the subject is easier than any of the topics in the second half of the old-time algebra, it is more interesting, and it admits of attractive outdoor work. It thus opens up a new field of interest for the pupil—the field of indirect measurement, in which there are discovered the first steps in the measuring of the distance to the stars and in the understanding of some of the former secrets of the universe in which we live. It follows naturally after the study of proportion, and its inclusion in algebra has now met with the approval of all leaders in the teaching of elementary mathematics. The initial work requires no knowledge of logarithms, a subject that may properly be left to a later course in algebra, simply because the time is hardly sufficient to allow for its introduction in the early stages of elementary algebra.

The fact that this topic has been recommended by the National Committee as part of elementary algebra, that it is required by the College Entrance Examination Board, that it is generally taught in close connection with algebra in other countries, that the plan has been generally approved by American associations of teachers, and that it has been followed in various recent textbooks assures its status in our elementary courses.

Much has been written of the advance in appreciation of the function concept in recent years. This advance is, of course, particularly noticeable in algebra, and the topic of trigonom-

etry is the one in which it is most in evidence. It is also seen in the treatment of the formula and in the entire subject of variation as a part of ratio and of fractions. It has of late come to be looked upon as a kind of unifying principle running through all parts of algebra, and as such, when not too consciously forced into the language of science, has undoubted value.

VIII. THE PROGRESS OF GEOMETRY

As a scientifically organized part of mathematics geometry is the oldest of its branches. For this reason it has had a longer period in which to perfect itself. It is therefore looked upon as less capable of reform or improvement than algebra and arithmetic.

The last quarter of a century has shown, however, that as a school subject it is capable of improvement in the same spirit if not to the same extent as these other branches.

For one thing, the recent years have clearly differentiated between intuitive and demonstrative geometry. While this has always been recognized in a small way, as in the treatment of simple mensuration in arithmetic, it was not until the Cambridge meeting of the International Mathematical Congress in 1912 that intuitive geometry was brought prominently before the educational section of that organization and began to be seriously considered by bodies of teachers throughout the world. Since then it has come, in this country, to occupy a worthy place in all our courses for the junior high schools. This place is properly in the seventh and eighth school years and to some extent even earlier. The subject naturally precedes demonstrative geometry, and our schools have come generally to recognize that it has but little sanction in the latter and more mature branch.

Demonstrative geometry twenty-five years ago consisted of at least one year of plane geometry, following the course in algebra, and at least a half year in solid geometry. In most schools there was a good deal of memorizing of demonstrations and the original exercise still played an almost negligible part,

being, for many pupils, without either purpose or pleasure. A few teachers enlivened the work by applications of doubtful value, but on the whole it was generally looked upon as an intellectual grind.

The progress since that time has been steady and encouraging. Its nature may be summarized briefly as follows:

1. There has been a more definite recognition by the schools that the chief purpose of demonstrative geometry is to show the application of logic to the proof of mathematical statements. It therefore requires a maturity of mind hardly found before the tenth school year, although for purposes of information a little work in demonstration may properly be given to the abler pupils in the preceding grade.

2. Therefore the purpose of demonstrative geometry is not mensuration, this being sufficiently cared for in the work in intuitive geometry; its purpose is, in part, to demonstrate the truths already known intuitively. For this reason the work in the mensuration of the circle has little sanction in demonstrative geometry, the rules being already known from intuitive geometry and the demonstrations as given not being very satisfactory from the standpoint of logic. The subject is therefore no longer required in college entrance examinations or for high school graduation. The same is true as to the mensuration of the rectangle, the rectangular solid, the cylinder, the sphere, and the cone.

3. The number of demonstrated theorems, and especially of the corollaries, has been greatly reduced, the purpose being to retain only the basal propositions that are of most use in the demonstrations of the "originals." This has shifted the emphasis from the book proofs, which usually constituted all the geometry of a century ago and most of that of the last quarter of the nineteenth century, to the original exercises where it properly belongs. Recent textbooks have an amount of original work of a simpler character that was hardly imagined a generation back. Indeed, the older geometries may be compared to an algebra that had all its examples fully

worked out, and no exercises for the pupils. The purpose of "book propositions" in geometry is largely that of worked-out examples in algebra,—to set a model for the pupil and to furnish a basis for his original work.

4. The number of solved problems has been proportionately reduced quite as much as the number of demonstrated theorems. The simpler constructions with ruler and compasses are given in intuitive geometry and their demonstration is not of much value as compared with the demonstrations of the theorems, leading as they do to only a small number of exercises and depending chiefly upon two or three simple theorems.

5. The exercises have greatly increased in number, but they have decreased in difficulty. The increase is due, as already stated, to the shifting of emphasis from that which an author has thought out for the pupil to that which the pupil is to think out for himself. The decrease in difficulty has arisen from the fact that the ability of pupils can certainly not be said to have increased during the period in which the schools have tended to the education of everyone rather than to that of a selected body of pupils of high intellectual promise. The tendency toward some form of universal high-school education is probably for the happiness of the race and the strengthening of the state, so that we shall have to accept, for many pupils, this lower standard.

There has, however, been another reason,—the feeling that a large number of simple exercises trains the immature pupil better than a small number of difficult ones. In our efforts to conform to this belief we are still in the experimental stage. The pupil of mathematical inclinations will prefer a more difficult type, and for him it will probably be better to pass rapidly over a few of the easy exercises and to come as soon as possible to those requiring more thought.

6. The discussion and generalization of propositions now holds higher place than it did a score of years ago; at least it is rather more in evidence in our courses of study and in our textbooks.

7. There is a strong movement on foot to cover the essential parts of plane and solid geometry in a single year. This is often met by the assertion that it is impossible. This assertion, however, depends upon the meaning assigned to the expression "essential parts." It would be feasible to frame a course in plane geometry that would require three years of hard work, but it would not lead to the most profitable use of the pupil's time. If we eliminate most of the construction problems, assume all the work in inequalities, eliminate all mention of incommensurables as applied to line segments and the circle, all the theory of proportion (treated of in algebra), the work in the mensuration of solids, and the rather purposeless treatment of spherical triangles, we can readily frame a very satisfactory course for a single year. This can be done by making selections from any standard geometry.

No mention has been made of the efforts toward developing courses in general mathematics. These refer rather to the method of presentation than to the improvement in subject matter. It may, however, be said that the recent development of the junior high school affords a natural field for combining different parts of mathematics in a single course, and this has been recognized in all our modern textbooks on the subject. Numerical trigonometry, also, naturally blends with algebra, and this is recognized in the recent courses of study. Demonstrative geometry, however, offers a different problem. It can use the algrebraic equation in its proofs, although it can get on about as well without it; but neither algebra nor trigonometry makes use of the demonstrations of geometry in its work. Our successful courses in general mathematics, therefore, tend to segregate demonstrative geometry, and this, psychologically, will have to be the case in the future. Either demonstrative geometry must be considered largely by itself or else it will tend to drop out of the curriculum or, at the best, to remain as a feeble memory of the world's effort to show how truth is logically established in the mathematical sciences.

IX. CONCLUSION

The progress of mathematics in our schools in the last quarter of a century may, then, be summarized briefly as follows:

1. Early attempts at improving the courses were greatly hampered by the force of tradition.

2. The most potent of the later influences for betterment have been the work of the International Commission on the Teaching of Mathematics; that of the National Committee on Mathematical Requirements; that of the College Entrance Examination Board, which brought together the secondary schools and the colleges; the rise of the junior high schools; the work of the schools of education; the improvements in textbooks; and the general Spirit of the Times.

3. The results of these labors are seen in the setting forth with greater clearness the aims which should guide in the teaching of each branch of mathematics. This is one of the two greatest gains. It has led to the elimination of much obsolete or relatively valueless material in arithmetic and algebra, to the introduction of new topics in each, to the merging of the first course in numerical trigonometry with the work in elementary algebra, to the elimination from geometry of matter of doubtful value, and to the general union of related parts of mathematics through such coordinating influences as that of the function concept and that of the social needs of our people.

4. The second of the gains of greatest importance has been the recognition of the rights of children to see the purposes of their studies, to find that the subjects synchronize with the development of their intellectual capacities, and to enjoy the work in mathematics as they should enjoy their work in other lines of intellectual activity.

5. There has been a notable advance in the testing of pupils' abilities and achievements, hampered only by the fact that many of the tests in algebra have tended to perpetuate some of the most objectionable features of the science,—a difficulty that will, of course, tend to disappear under the combined

efforts of those with some mathematical vision and those who know the technique of testing.

6. In no field of elementary or secondary education has advancement in the last twenty-five years been more marked than in that of mathematics. If teachers feel discouraged with the reactionary attitude of certain administrators or of boards of control in state or city, they may well take courage by considering the state of high school mathematics at the beginning of the century and comparing it with the state of the subject at the present time.

ON THE FOUNDATIONS OF MATHEMATICS*

By Eliakim Hastings Moore

The American Mathematical Society gives its retiring president the privilege of speaking on whatever he may have at heart. Accordingly, this afternoon I propose to consider with you some matters of importance—indeed, perhaps of fundamental importance—in the development of mathematics in this country; and it will duly appear in what non-technical sense I am speaking 'On the Foundations of Mathematics.'

A VIEW

Abstract Mathematics.—The notion within a given domain of defining the objects of consideration rather by a body of properties than by particular expressions or intuitions is as old as mathematics itself. And yet the central importance of the notion appeared only during the last century—in a host of researches on special theories and on the foundations of geometry and analysis. Thus has arisen the general point of view of what may be called *abstract mathematics.* One comes in touch with the literature very conveniently by the mediation of Peano's *Revue des Mathématiques.* The Italian school of Peano and the *Formulaire Mathématique,* published in connection with the *Revue,* are devoted to the codification in Peano's symbolic language of the principal mathematical theories, and to researches on abstract mathematics. General interest in abstract mathematics was aroused by Hilbert's Gauss-Weber Festschrift of 1899: 'Ueber die Grundlagen der Geometrie,' a memoir rich in results and suggestive in meth-

*Presidential address delivered before The American Mathematical Society at its ninth annual meeting, December 29, 1902. Reprinted here from *Science*, N. S., Vol. XVII, pp. 401-416, March 13, 1903; punctuation and other matters of style are as in the original.

ods; I refer to the reviews by Sommer,* Poincaré,† Halsted,‡ Hedrick§ and Veblen.‖

We have as a basal science logic, and as depending upon it the special deductive sciences which involve undefined symbols and whose propositions are not all capable of proof. The symbols denote either classes of elements or relations amongst elements. In any such science one may choose in various ways the system of undefined symbols and the system of undemonstrated or primitive propositions, or postulates. Every proposition follows from the postulates by a finite number of logical steps. A careful statement of the fundamental generalities is given by Padoa in a paper¶ before the Paris Congress of Philosophy, 1900.

Having in mind a definite system of undefined symbols and a definite system of postulates, we have first of all the notion of the compatibility of these postulates; that is, that it is impossible to prove by a finite number of logical steps the simultaneous validity of a statement and its contradictory statement; in the next place, the question of the independence of the postulate or the irreducibility of the system of postulates; that is, that no postulate is provable from the remaining postulates. Padoa introduces the notion of the irreducibility of the system of undefined symbols. A system of undefined symbols is said to be reducible if for one of the symbols, X, it is possible to establish, as a logical consequence of the assumption of the validity of the postulates, a nominal or symbolic definition of the form $X=A$, where in the expression A there enter only the undefined symbols distinct from X. For the purpose of practical application, it seems to be desirable to modify the definition so as to call the system of undefined symbols reducible if there is a nominal definition

**Bull. Amer. Math. Soc.* (2), vol. 6 (1900), p. 287.
†*Bull. Sciences Mathém.*, vol. 26 (1902), p. 249.
‡*The Open Court*, September, 1902.
§*Bull. Amer. Math. Soc.* (2), vol. 9 (1902), p. 158.
‖*The Monist*, January, 1903.
¶'Essai d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque,' *Bibliothèque du Congrès International de Philosophie*, vol. 3, p. 309.

$X=A$ of one of them X in terms of the others such that in any interpretation of the science the postulates retain their validity when instead of the initial interpretation of the symbol X there is placed the interpretation of the symbol A of that symbol. If the system of symbols is reducible in the sense of the original definition it is in the sense of the new definition, but not necessarily conversely, as appears for instance from the following example, occurring in the foundations of geometry.

Hilbert uses the following undefined symbols: 'point,' 'line,' 'plane,' 'incidence' of point and line, 'incidence' of point and plane, 'between,' and 'congruent.' Now it is possible to give for the symbol 'plane' a symbolic definition in terms of the other undefined symbols—for instance, a plane is a certain class of points (as Peano showed in 1892), or again, a plane is a certain class of lines; while the notion 'incidence' of point and plane receives convenient definition. It is apparent from the fact that these definitions may be given in these two ways that Hilbert's system of undefined symbols is not in Padoa's sense irreducible, at least, in so far as the symbols 'plane,' 'incidence' of point and plane are concerned—while it is equally clear that these symbols are in the abstract geometry superfluous.

In his dissertation on Euclidean geometry, Mr. Veblen, following the example of Pasch and Peano, takes as undefined symbols 'point' and 'between,' or 'point' and 'segment.' In terms of these two symbols alone he expresses a set of independent fundamental postulates of Euclidean geometry, in the first place developing the projective geometry, and then as to congruence relating himself to the point of view of Klein in his 'Erlangen Programm,' whereby the group of movements of Euclidean geometry enters as a certain subgroup of the group of collineations of projective geometry. Here arises an interesting question as to the sense in which the undefined symbol 'congruence' is superfluous in the Euclidean geometry based upon the symbols 'point,' 'between.' One sees at once that a definition of 'congruence' involves parametric points in

its expression, while on the other hand a definition of the system of all 'planes,' that is, of the general concept 'plane,' involves no such parametric elements. But, again, just as there exist distinct definitions of 'congruence,' owing to a variation of the parametric points, so there exist distinct definitions of the general concept 'plane,' as was indicated a moment ago. One has the feeling that the state of affairs must be as follows: In any interpretation of, say, Hilbert's symbols, wherein the postulates of Hilbert are valid, every valid statement which does not involve the symbol 'plane' in direct connection with the general logical symbol (=) of symbolic definition, remains valid when we modify it in accordance with either of the definitions of 'plane' previously referred to. On the other hand, this state of affairs does not hold for the symbol 'congruence.' The proof of the former statement would seem to involve fundamental logical niceties.

The compatibility and the independence of the postulates of a system of postulates of a special deductive science have been up to this time always made to depend upon the self-consistency of some other deductive science; for instance, geometry depends thus upon analysis, or analysis upon geometry. The fundamental and still unsolved problem in this direction is that of the direct proof of the compatibility of the postulates of arithmetic, or of the real number system of analysis. (To the society this morning Dr. Huntington exhibited two sets of independent postulates for this real number system.) This is the second of the twenty-three problems listed by Hilbert in his address before the Paris Mathematical Congress of 1900.

The Italian writers on abstract mathematics for the most part make use of Peano's symbolism. One may be tempted to feel that this symbolism is not an essential part of their work. It is only right to state, however, that the symbolism is not difficult to learn, and that there is testimony to the effect that the symbolism is actually of great value to the investigator in removing from attention the concrete connotations of the ordinary terms of general and mathematical language. But

of course the essential difficulties are not to be obviated by the use of any symbolism, however delicate.

Indeed the question arises whether the abstract mathematicians in making precise the metes and bounds of logic and the special deductive sciences are not losing sight of the evolutionary character of all life-processes, whether in the individual or in the race. Certainly the logicians do not consider their science as something now fixed. All science, logic and mathematics included, is a function of the epoch—all science, in its ideals as well as in its achievements. Thus with Hilbert let a special deductive or mathematical science be based upon a finite number of symbols related by a finite number of compatible postulates, every proposition of the science being deducible by a finite number of logical steps from the postulates. The content of this conception is far from absolute. It involves what presuppositions as to general logic? What is a finite number? In what sense is a postulate—for example, that any two distinct points determine a line—a single postulate? What are the permissible logical steps of deduction? Would the usual syllogistic steps of formal logic suffice? Would they suffice even with the aid of the principle of mathematical induction, in which Poincaré finds* the essential synthetic element of mathematical argumentation the basis of that generality without which there would be no science? In what sense is mathematical induction a single logical step of deduction?

One has then the feeling that the carrying out in an absolute sense of the program of the abstract mathematicians will be found impossible. At the same time, one recognizes the importance attaching to the effort to do precisely this thing. The requirement of rigor tends toward essential simplicity of procedure, as Hilbert has insisted in his Paris address, and the remark applies to this question of mathematical logic and its abstract expression.

Pure and Applied Mathematics.—In the ultimate analysis

*'Sur la nature du raisonnement mathématique,' *Revue de Métaphysique et de Morale*, vol. 2 (1894), pp. 371-384.

for any epoch, we have general logic, the mathematical sciences,† that is, all special formally and abstractly deductive self-consistent sciences, and the natural sciences, which are inductive and informally deductive. While this classification may be satisfactory as an ideal one, it fails to recognize the fact that in mathematical research one by no means confines himself to processes which are mathematical according to this definition; and if this is true with respect to the research of professional mathematicians, how much more is it true with respect to the study, which should throughout be conducted in the spirit of research, on the part of students of mathematics in the elementary schools and colleges and universities. I refer to the articles* of Poincaré on the role of intuition and logic in mathematical research and education.

It is apparent that this ideal classification can be made by the devotee of science only when he has reached a considerable degree of scientific maturity, that perhaps it would fail to appeal to non-mathematical experts, and that it does not accord with the definitions given by practical work in mathematicians. Indeed, the attitude of practical mathematicians toward this whole subject of abstract mathematics, and especially the symbolic form of abstract mathematics, is not unlike that of the practical physicist toward the whole subject of theoretic mathematics, and in turn not unlike that of the practical engineer toward the whole subject of theoretical physics and mathematics. Furthermore, every one understands that many of the most important advances of pure mathematics have arisen in connection with investigations originating in the domain of natural phenomena.

Practically then it would seem desirable for the interests of science in general that there should be a strong body of men

†Of which none is at present known to exist.

*'La logique et l'intuition dans la science mathématique et dans l'enseignement,' *L'Enseignement Mathématique*, vol. 1 (1899), pp. 157-162. 'Du role de l'intuition et de la logique en mathématiques,' *Compte Rendu du Deuxième Congrès International des Mathématiciens, Paris* (1900), 1902, pp. 115-130. 'Sur les rapports de l'analyse pure et de la physique mathématique,' Conference, Zurich, 1897; *Acta Mathematica*, vol. 21, p. 238.

thoroughly possessed of the scientific method in both its inductive and its deductive forms. We are confronted with the questions: What is science? What is the scientific method? What are the relations between the mathematical and the natural scientific processes of thought? As to these questions, I refer to articles and addresses of Poincaré,* Boltzmann† and Burkhardt,‡ and to Mach's 'Science of Mechanics' and Pearson's 'Grammar of Science.'

Without elaboration of metaphysical or psychological details, it is sufficient to refer to the thought that the individual, as confronted with the world of phenomena in his effort to obtain control over this world, is gradually forced to appreciate a knowledge of the usual coexistences and sequences of phenomena, and that science arises as the body of formulas serving to epitomize or summarize conveniently these usual coexistences and sequences. These formulas are of the nature of more or less exact descriptions of phenomena; they are not of the nature of explanations. Of all the relations entering into the formulas of science, the fundamental mathematical notions of number and measure and form were among the earliest, and pure mathematics in its ordinary acceptation may be understood to be the systematic development of the properties of these notions, in accordance with conditions prescribed by physical phenomena. Arithmetic and geometry, closely united in mensuration and trigonometry, early reached a high degree of advancement. But after the development of the generalizing literal notation of algebra, and largely in response to the insistent demands of mechanics, astronomy and physics, the seventeenth century, binding together arithmetic and geometry infinitely more closely, created analytic geometry and the in-

*In addition to those already cited: 'On the Foundations of Geometry,' *The Monist*, vol. 9, October, 1898, pp. 1-43. 'Sur les principes de la mécanique,' *Bibliothèque du Congrès International de Philosophie*, vol 3, pp. 457-494.

†'Ueber die Methoden der theoretischen Physik,' *Dyck's Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente*, pp. 89-98, Munich, 1892.

‡'Mathematisches und Naturwissenschaftliches Denken,' *Jahresbericht der Deutschen Math.-Ver.*, vol. 11 (1902), pp. 49-57.

finitesimal calculus, those mighty methods of research whose application to all branches of the theoretical and practical physical sciences so fundamentally characterizes the civilization of to-day.

The eighteenth century was devoted to the development of the powers of these new instruments in all directions. While this development continued during the nineteenth century, the dominant note of the nineteenth century was that of critical reorganization of the foundations of pure mathematics, so that, for instance, the majestic edifice of analysis was seen to rest upon the arithmetic of positive integers alone. This reorganization and the consequent course of development of pure mathematics were independent of the question of the application of mathematics to the sister sciences. There has thus arisen a chasm between pure mathematics and applied mathematics. There have not been lacking, however, influences making toward the bridging of this chasm; one thinks especially of the whole influence of Klein in Germany and of the École Polytechnic in France. As a basis of union of the pure mathematicians and the applied mathematicians, Klein has throughout emphasized the importance of a clear understanding of the relations between those two parts of mathematics which are conveniently called 'mathematics of precision' and 'mathematics of approximation,' and I refer especially to his latest work of this character, 'Anwendung der Differential und Integral-Rechnung auf Geometrie: Eine Revision der Principien' (Göttingen, summer semester, 1901, Teubner, 1902). This course of lectures is designed to present particular applications of the general notions of Klein, and furthermore, it is in continuation of the discussion between Pringsheim and Klein and others, as to the desirable character of lectures on mathematics in the universities of Germany.

Elementary Mathematics.—This separation between pure mathematics and applied mathematics is grievous even in the domain of elementary mathematics. In witness, in the first place: The workers in physics, chemistry and engineering need more practical mathematics; and numerous textbooks, in par-

ticular, on calculus, have recently been written from the point of view of these allied subjects. I refer to the works by Nernst and Schoenflies,* Lorentz,† Perry,‡ and Mellor,§ and to a book on the very elements of mathematics now in preparation by Oliver Lodge.

In the second place, I dare say you are all familiar with the surprisingly vigorous and effective agitation with respect to the teaching of elementary mathematics which is at present in progress in England, largely under the direction of John Perry, professor of mechanics and mathematics of the Royal College of Science, London, and chairman of the Board of Examiners of the Board of Education in the subjects of engineering, including practical plane and solid geometry, applied mechanics, practical mathematics, in addition to more technical subjects, and in this capacity in charge of the education of some hundred thousand apprentices in English night schools. The section on Education of the British Association had its first session at the Glasgow meeting, 1901, and the session was devoted to the consideration, in connection with the section on Mathematics and Physics, of the question of the pedagogy of mathematics, and Perry opened the discussion by a paper on 'The Teaching of Mathematics.' A strong committee under the chairmanship of Professor Forsyth, of Cambridge, was appointed 'to report upon improvements that might be effected in the teaching of mathematics, in the first instance, in the teaching of elementary mathematics, and upon such means as they think likely to effect such improvements.' The paper of Perry, with the discussion of the subject at Glasgow,

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*Nernst und Schoenflies, 'Einführung in die mathematische Behandlung der Naturwissenschaften' (Munich and Leipsic, 1895); the basis of Young and Linebarger's 'Elements of Differential and Integral Calculus,' New York, 1900.

†Lorentz, 'Lehrbuch der Differential- und Integralrechnung,' Leipsic, 1900.

‡Perry, 'Calculus for Engineers' (second edition, London, E. Arnold, 1897); German translation by Fricke (Teubner, 1902). Cf. also the citations given later on.

§Mellor, 'Higher Mathematics for Students of Chemistry and Physics, with special reference to Practical Work,' Longmans, Green & Co., 1902, pp. xxi+543.

and additions including the report of the committee as presented to the British Association at its Belfast meeting, September, 1902, are collected in a small volume, 'Discussion on the Teaching of Mathematics,' edited by Professor Perry (Macmillan, second edition, 1902).*

One should consult the books of Perry, 'Practical Mathematics,'† 'Applied Mechanics,'‡ 'Calculus for Engineers'§ and 'England's Neglect of Science,'‖ and his address* on 'The Education of Engineers'—and furthermore the files from 1899 on of the English journals, *Nature, School World, Journal of Education* and *Mathematical Gazette.*

One important purpose of the English agitation is to relieve the English secondary school teachers from the burden of a too precise examination system, imposed by the great examining bodies; in particular, to relieve them from the need of retaining Euclid as the sole authority in geometry, at any rate with respect to the sequence of propositions. Similar efforts made in England about thirty years ago were unsuccessful. Apparently the forces operating since that time have just now broken forth into successfull activity; for the report of the British Association committee was distinctly favorable, in a conservative sense, to the idea of reform, and already noteworthy initial changes have been made in the regulations for the secondary examinations by the examination syndicates of the universities of Oxford, Cambridge, and London.

The reader will find the literature of this English movement very interesting and suggestive. For instance, in a letter to *Nature* (vol. 65, p. 484, March 27, 1902) Perry mildly apologizes for having to do with the movement whose immediate results

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*Cf. also 'Report on the Teaching of Elementary Mathematics issued by the Mathematical Association,' G. Bell & Sons, London, 1902.

†Published for the Board of Education by Eyre and Spottiswoode, London, 1899.

‡D. Van Nostrand Co., New York, 1898.

§Second edition, London, E. Arnold, 1897.

‖T. Fisher Unwin, London, 1900.

*In opening the discussion of the sections on Engineering and on Education at the Belfast, 1902, meeting of the British Association; published in SCIENCE, November 14, 1902.

are likely to be merely slight reforms, instead of thoroughgoing reforms called for in his pronouncements and justified by his marked success during over twenty years as a teacher of practical mathematics. He asserts that the orthodox logical sequence in mathematics is not the only possible one; that, on the contrary, a more logical sequence than the orthodox one (because one more possible of comprehension by students) is based upon the notions underlying the infinitesimal calculus taken as axioms; for instance, that a map may be drawn to scale; the notions underlying the many uses of squared paper; that decimals may be dealt with as ordinary numbers. He asserts as essential that the boy should be *familiar* (by way of experiment, illustration, measurement, and by every possible means) with the ideas to which he applies his logic; and moreover that he should be thoroughly *interested* in the subject studied; and he closes with this peroration:

> "'Great God! I'd rather be
> A pagan, suckled in a creed outworn.'

I would rather be utterly ignorant of all the wonderful literature and science of the last twenty-four centuries, even of the wonderful achievements of the last fifty years, than not to have the sense that our whole system of so-called education is as degrading to literature and philosophy as it is to English boys and men."

As a pure mathematician, I hold as the most important suggestion of the English movement the suggestion of Perry's, just cited, that by emphasizing steadily the practical sides of mathematics, that is, arithmetic computations, mechanical drawing and graphical methods generally, in continuous relation with problems of physics and chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry, and the calculus. This is accomplished, on the one hand, by the increase of attention and comprehension obtained by connecting the abstract mathematics with subjects which are naturally of interest to the boy, so that, for instance, all the results obtained by theoretic process are capable of check by

laboratory process, and, on the other hand, a diminution of emphasis on the systematic and formal sides of the instruction in mathematics. Undoubtedly many mathematicians will feel that this decrease of emphasis will result in much, if not irreparable, injury to the interests of mathematics. But I am inclined to think that the mathematician with the catholic attitude of an adherent of science, in general (and at any rate with respect to the problems of the pedagogy of elementary mathematics there would seem to be no other rational attitude) will see that the boy will be learning to make practical use in his scientific investigations—to be sure, in a naive and elementary way—of the finest mathematical tools which the centuries have forged; that under skilful guidance he will learn to be interested not merely in the achievements of the tools, but in the theory of the tools themselves, and that thus he will ultimately have a feeling towards his mathematics extremely different from that which is now met with only too frequently—a feeling that mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions.

The American Mathematical Society.—The American Mathematical Society has, naturally, interested itself chiefly in promoting the interests of research in mathematics. It has, however, recognized that those interests are closely bound up with the interests of education in mathematics. I refer in particular to the valuable work done by the committee appointed, with the authorization of the Council, by the Chicago section of the society, to represent mathematics in connection with Dr. Nightingale's committee of 1899 of the National Educational Association in the formulation of standard curricula for high schools and academies, and to the fact that two committees are now at work, one appointed in December, 1901, by the Chicago Section, to formulate the desirable conditions for the granting, by institutions of the Mississippi valley, of the degree of Master of Arts for work in mathematics, and the other appointed by the society at its last summer meeting to cooperate with similar committees of the National Education-

al Association and of the Society for the Promotion of Engineering Education, in formulating standard definitions of requirements in mathematical subjects for admission to colleges and technological schools; and furthermore I refer to the fact that (although not formally) the society has made a valuable contribution to the interests of secondary education in that the College Entrance Examination Board has as its secretary the principal founder of the society. I have accordingly felt at liberty to bring to the attention of the society these matters of pedagogy of elementary mathematics, and I do so with the firm conviction that it would be possible for the society, by giving still more attention to these matters, to further most effecively the highest interests of mathematics in this country.

A VISION

An Invitation.—The pure mathematicians are invited to determine how mathematics is regarded by the world at large, including their colleges of other science departments and the students of elementary mathematics, and to ask themselves whether by modification of method and attitude they may not win for it the very high position in general esteem and appreciative interest which it assuredly deserves.

This general invitation and the preceding summary view invoke this vision of the future of elementary mathematics in this country.

The Pedagogy of Elementary Mathematics.—We survey the pedagogy of elementary mathematics in the primary schools, in the secondary schools and in the junior colleges (the lower collegiate years.) It is, however, understood that there is a movement for the enlargement of the strong secondary schools, by the addition of the two years of junior college work and by the absorption of the last two or three grades of the primary schools, into institutions more of the type of the German gymnasia and the French lycée;* in favor of this movement

*As to the mathematics of these institutions, one may consult the book on 'The Teaching of Mathematics in the Higher School of Prussia' (New York, Longmans, Green & Co., 1900) by Professor Young, and the article (*Bulletin Amer. Math. Soc.* (2), vol. 6, p. 225) by Professor Pierpont.

there are strong arguments, and among them this, that in such institutions, especially if closely related to strong colleges or universities, the mathematical reforms may the more easily be carried out.

The fundamental problem is that of *the unification of pure and applied mathematics.* If we recognize the branching implied by the very terms 'pure,' 'applied,' we have to do with a special case of *the correlation of different subjects* of the curriculum, a central problem in the domain of pedagogy from the time of Herbart on. In this case, however, the fundamental solution is to be found rather by way of indirection—by arranging the curriculum so that throughout the domain of elementary mathematics the branching be not recognized.

The Primary Schools.—Would it not be possible for the children in the grades to be trained in power of observation and experiment and reflection and deduction so that always their mathematics should be directly connected with matters of thoroughly concrete character? The response is immediate that this is being done to-day in the kindergartens and in the better elementary schools. I understand that serious difficulties arise with children of from nine to twelve years of age, who are no longer contented with the simple, concrete methods of earlier years and who, nevertheless, are unable to appreciate the more abstract methods of the later years. These difficulties, some say, are to be met by allowing the mathematics to enter only implicitly in connection with the other subjects of the curriculum. But rather the material and methods of the mathematics should be enriched and vitalized. In particular, the grade teachers must make wiser use of the foundations furnished by the kindergarten. The drawing and the paper folding must lead on directly to systematic study of intuitional geometry,* including the construction of models and the elements of mechanical drawing, with simple exercises in geo-

*Here I refer to the very suggestive paper of Benchara Branford, entitled 'Measurement and Simple Surveying. An Experiment in the Teaching of Elementary Geometry' to a small class of beginners of about ten years of age (*Journal of Education*, London, the first part appearing in the number for August, 1899.)

metrical reasoning. The geometry must be closely connected with the numerical and literal arithmetic. The cross-grooved tables of the kindergarten furnish an especially important type of connection, viz., a conventional graphical depiction of any phenomenon in which one magnitude depends upon another. These tables and the similar cross-section blackboards and paper must enter largely into all the mathematics of the grades. The children are to be taught to represent, according to the usual conventions, various familiar and interesting phenomena and to study the properties of the phenomena in the pictures: to know, for example, what concrete meaning attaches to the fact that a graph curve at a certain point is going down or going up or is horizontal. Thus the problems of percentage—interest, etc.—have their depiction in straight or broken line graphs.

The Secondary Schools.—Pending the reform of the primary schools, the secondary schools must advance independently. In these schools at present, according to one type of arrangement, we find algebra in the first year, plane geometry in the second, physics in the third, and the more difficult parts of algebra and solid geometry, with review of all the mathematics in the fourth.

Engineers* tell us that in the schools algebra is taught in one water-tight compartment, geometry in another, and physics

*Why is it that one of the sanest and best-informed scientific men living, a man not himself an engineer, can charge mathematicians with killing off every engineering school on which they can lay hands? Why do engineers so strongly urge that the mathematical courses in engineering schools be given by practical engineers?

And why can a reviewer of 'Some Recent Books of Mechanics' write with truth: "The students' previous training in algebra, geometry, trigonometry, analytic geometry and calculus as it is generally taught has been necessarily quite formal. These mighty algorithms of formal mathematics must be learned so that they can be applied with readiness and precision. But with mechanics comes the application of these algorithms, and formal, do-by-rote methods, though often possible, yield no results of permanent value. How to elicit and cultivate thought is now of primary importance"? (E. B. Wilson, *Bulletin Amer. Math. Soc.*, October, 1902.) But is it conceivable that in any part of the education of the student the problem of eliciting and cultivating thought should not be of primary importance?

in another, and that the student learns to appreciate (if ever) only very late the absolutely close connection between these different subjects, and then, if he credits the fraternity of teachers with knowing the closeness of this relation, he blames them most heartily for their unaccountably stupid way of teaching him. If we contrast this state of affairs with the state of affairs in the solid four years' course in Latin, I think we are forced to the conclusion that the organization of instruction in Latin is much more perfect than that of the instruction in mathematics.

The following question arises: *Would it not be possible to organize the algebra, geometry and physics of the secondary school into a thoroughly coherent four years' course,* comparable in strength and closeness of structure with the four years' course in Latin? (Here under physics I include astronomy, and the more mathematical and physical parts of physiography.) It would seem desirable that, just as the systematic development of theoretical mathematics is deferred to a later period, likewise much of theoretical physics might well be deferred. Let the physics also be made thoroughly practical. At any rate, so far as the instruction of boys is concerned, the course should certainly have its character largely determined by the conditions which would be imposed by engineers. What kind of two or three years' course in mathematics and physics would a thoroughly trained engineer give to boys in the secondary school? Let this body of material postulated by the engineer serve as the basis of the four years' course. Let the instruction in the course, however, be given by men who have received expert training in mathmatics and physics as well as in engineering and let the instruction be so organized that with the development of the boy, in appreciation of the practical relations, shall come simultaneously his development in the direction of theoretical physics and theoretical mathematics.

Perry is quite right in insisting that it is scientifically legitimate in the pedagogy of elementary mathematics to take a large body of basal principles instead of a small body and to

build the edifice upon the larger body for the earlier years, reserving for the later years the philosophic criticism of the basis itself and the reduction of the basal system.

To consider the subject of geometry in all briefness: with the understanding that proper emphasis is laid upon all the concrete sides of the subject, and that furthermore from the beginning exercises in informal deduction* are introduced increasingly frequently, when it comes to the beginning of the more formal deductive geometry why should not the students be directed each for himself to set forth a body of geometric fundamental principles, on which he would proceed to erect his geometric edifice? This method would be thoroughly practical and at the same time thoroughly scientific. The various students would have different systems of axioms, and the discussions thus arising naturally would make clearer in the minds of all precisely what are the functions of the axioms in the theory of geometry. The students would omit very many of the axioms, which to them would go without saying. The teacher would do well not to undertake to make the system of axioms thoroughly complete in the abstract sense. "Sufficient unto the day is the precision thereof." The student would very probably wish to take for granted all the ordinary properties of measurement and of motion, and would be ready at once to accept the geometrical implications of coordinate geometry. He could then be brought with extreme ease to the consideration of fundamental notions of the calculus as treated concretely, and he would find those notions delightfully real and powerful, whether in the domain of mathematics or of physics or of chemistry.

To be sure, as Study has well insisted, for a thorough comprehension of even the elementary parts of Euclidean geometry the non-Euclidean geometries are absolutely essential. But

*In an article shortly to appear in the *Educational Review*, on 'The Psychological and the Logical in the Teaching of Geometry,' Professor John Dewey, calling attention to the evolutionary character of the education of an individual, insists that there should be no abrupt transition from the introductory, intuitional geometry to the systematic, demonstrative geometry.

the teacher is teaching the subject for the benefit of the students, and it must be admitted that beginners in the study of demonstrative geometry can not appreciate the very delicate considerations involved in the thoroughly abstract science. Indeed, one may conjecture that, had it not been for the brilliant success of Euclid in his effort to organize into a formally deductive system the geometric treasures of his times, the advent of the reign of science in the modern sense might not have been so long deferred. Shall we then hold that in the schools the teaching of demonstrative geometry should be reformed in such a way as to take account of all the wonderful discoveries which have been made—many even recently—in the domain of abstract geometry? And should similar reforms be made in the treatment of arithmetic and algebra? To make reforms of this kind, would it not be to repeat more gloriously the error of those followers of Euclid who fixed his 'Elements' as a textbook for elementary instruction in geometry for over two thousand years? Every one agrees that professional mathematicians should certainly take account of these great developments in the technical foundations of mathematics, and that ample provision should be made for instruction in these matters; and on reflection, every one agrees further that this provision should be reserved for the later collegiate and university years.

The Laboratory Method.—This program of reform calls for the development of a thoroughgoing laboratory system of instruction in mathematics and physics, a principal purpose being as far as possible to develop on the part of every student the true spirit of research, and an appreciation, practical as well as theoretic, of the fundamental methods of science.

In connection with what has already been said, the general suggestions I now add will, I hope, be found of use when one enters upon the questions of detail involved in the organization of the course.

As the world of phenomena receives attention by the individual, the phenomena are described both graphically and in terms of number and measure; the number and measure re-

lations of the phenomena enter fundamentally into the graphical depiction, and furthermore the graphical depiction of the phenomena serves powerfully to illuminate the relations of number and measure. This is the fundamental scientific point of view. Here under the term graphical depiction I include representation by models.

To provide for the needs of laboratory instruction, there should be regularly assigned to the subject two periods, counting as one period in the curriculum.

As to the possibility of effecting this unification of mathematics and physics in the secondary schools, objection will be made by some teachers that it is impossible to do well more than one thing at a time. This pedagogic principle of concentration is undoubtedly sound. One must, however, learn how to apply it wisely. For instance, in the physical laboratory it is undesirable to introduce experiments which teach the use of the calipers or of the vernier or of the slide rule. Instead of such uninteresting experiments of limited purpose, the students should be directed to extremely interesting problems which involve the use of these instruments, and thus be led to learn to use the instruments as a matter of course, and not as a matter of difficulty. Just so the smaller elements of mathematical routine can be made to attach themselves to laboratory problems, arousing and retaining the interest of the students. Again, everything exists in its relations to other things, and in teaching the one thing the teacher must illuminate these relations.

Every result of importance should be obtained by at least two distinct methods, and every result of especial importance by two essentially distinct methods. This is possible in mathematics and the physical sciences, and thus the student is made thoroughly independent of all authority.

All results should be checked, if only qualitatively or if only 'to the first significant figure.' In setting problems in practical mathematics (arithmetical computation or geometrical construction) the teacher should indicate the amount or percentage of error permitted in the final result. If this amount

of percentage is chosen conveniently in the different examples, the student will be led to the general notion of closer and closer approximation to a perfectly definite result, and thus in a practical way to the fundamental notions of the theory of limits and of irrational numbers. Thus, for instance, uniformity of convergence can be taught beautifully in connection with the concrete notion of area under a monotonic curve between two ordinates, by a figure due to Newton, while the interest will be still greater if in the diagram area stands for work done by an engine.

The teacher should lead up to an important theorem gradually in such a way that the precise meaning of the statement in question, and further, the practical, *i. e.*, computational or graphical or experimental—truth of the theorem is fully appreciated; and, furthermore, the importance of the theorem is understood, and, indeed, the desire for the formal proof of the proposition is awakened, before the formal proof itself is developed. Indeed, in most cases, much of the proof should be secured by the research work of the students themselves.

Some hold that absolutely individual instruction is the ideal, and a laboratory method has sometimes been used for the purpose of attaining this ideal. The laboratory method has as one of its elements of great value the flexibility which permits students to be handled as individuals or in groups. The instructor utilizes all the experience and insight of the whole body of students. He arranges it so that the students consider that they are studying the subject itself, and not the words, either printed or oral, of any authority on the subject. And in this study they should be in the closest cooperation with one another and with their instructor, who is in a desirable sense one of them and their leader. Instructors may fear that the brighter students will suffer if encouraged to spend time in cooperation with those not so bright. But experience shows that just as every teacher learns by teaching, so even the brightest students will find

themselves much the gainers for this cooperation with their colleagues.

In agreement with Perry, it would seem possible that the student might be brought into vital relation with the fundamental elements of trigonometry, analytic geometry and the calculus, on condition that the whole treatment in its origin is and in its development remains closely associated with thoroughly concrete phenomena. With the momentum of such practical education in the methods of research in the secondary school, the college students would be ready to proceed rapidly and deeply in any direction in which their personal interests might lead them. In particular, for instance, one might expect to find effective interest on the part of college students in the most formal abstract mathematics.

For all students who are intending to take a full secondary school course in preparation for colleges or technological schools, I am convinced that the laboratory method of instruction in mathematics and physics, which has been briefly suggested, is the best method of instruction—for students in general, and for students expecting to specialize in pure mathematics, in pure physics, in mathematical physics or astronomy, or in any branch of engineering.

Evolution, not Revolution.—In contemplating this reform of secondary school instruction we must be careful to remember that it is to be accomplished as an evolution from the present system, and not as a revolution of that system. Even under the present organization of the curriculum the teachers will find that much improvement can be made by closer cooperation one with another; by the introduction, so far as possible, of the laboratory two-period plan; and in any event by the introduction of laboratory methods; laboratory record books, cross-section paper, computational and graphical methods in general, including the use of colored inks and chalks; the cooperation of students; and by laying emphasis upon the comprehension of propositions rather than upon the exhibition of comprehension.

The Junior Colleges.—Just as secondary schools should begin to reform without waiting for the improvement of the primary schools, so the elementary collegiate courses should be modified at once without waiting for the reform of the secondary schools. And naturally, in the initial period of reform, the education in each higher domain will involve many elements which later on will be transferred to a lower domain.

Further, by the introduction into the junior colleges of the laboratory method of instruction it will be possible for the colleges and universities to take up a duty which for the most part has been neglected in this country. For, although we have normal schools and other training schools for those who expect to teach in the grades, little attention has as yet been given to the training of those who will become secondary school teachers. The better secondary schools of to-day are securing the services of college graduates who have devoted special attention to the subjects which they intend to teach, and as time goes on the positions in these schools will as a rule be filled (as in France and Germany) by those who have supplemented their college course by several years of university work. Here these college and university graduates proceed at once to their work in the secondary schools. Now in the laboratory courses of the junior college, let those students of the senior college and graduate school who are to go into the teaching career be given training in the pedagogy of mathematics according to the laboratory system; for such a student the laboratory would be a laboratory in the pedagogy of mathematics; that is, he would be a colleague-assistant of the instructor. By this arrangement, the laboratory instruction of the colleges would be strengthened at the same time that well equipped teachers would be prepared for work in the secondary schools.

The Freedom of the Secondary Schools.—The secondary schools are everywhere preparing students for colleges and technological schools, and whether the requirements of those institutions are expressed by way of examination of students

or by way of the conditions for the accrediting of schools or teachers, the requirements must be met by the secondary schools. The stronger secondary school teachers too often find themselves shackled by the specific requirements imposed by local or college authorities. Teaching must become more of a profession. And this implies not only that the teacher must be better trained for his career, but also in his career he be given with greater freedom greater responsibility. To this end closer relations should be established between the teachers of the colleges and those of the secondary schools; standing provisions should be made for conferences as to improvement of the secondary school curricula and in the collegiate admission requirements; and the leading secondary school teachers should be steadily encouraged to devise and try out plans looking in any way toward improvement.

Thus the proposed four years' laboratory course in mathematics and physics will come into existence by way of evolution. In a large secondary school, the strongest teachers, finding the project desirable and feasible, will establish such a course alongside the present series of disconnected courses—and as time goes on their success will in the first place stimulate their colleagues to radical improvements of method under the present organization and finally to a complete reorganization of the courses in mathematics and physics.

The American Mathematical Society.—Do you not feel with me that the American Mathematical Society, as the organic representative of the highest interests of mathematics in this country, should be directly related with the movement of reform? And, to this end, that the society, enlarging its membership by the introduction of a large body of the strongest teachers of mathematics in the secondary schools, should give continuous attention to the question of improvement of education in mathematics, in institutions of all grades? That there is need for the careful consideration of such questions by the united body of experts, there is no doubt whatever, whether or not the general suggestions which we have been considering this afternoon turn out to be desirable and practicable. In

case the question of pedagogy does come to be an active one, the society might readily hold its meetings in two divisions—a division of research and a divison of pedagogy.

Furthermore, there is evident need of a national organization having its center of gravity in the whole body of science instructors in the secondary schools; and those of us interested in these questions will naturally relate ourselves also to this organization. It is possible that the newly formed Central Association of Physics Teachers may be the nucleus of such an organization.

CONCLUSION

The successful execution of the reforms proposed would seem to be of fundamental importance to the development of mathematics in this country. I urge that individuals and organizations proceed to the consideration of the general question of reform with all the related questions of detail. Undoubtedly in many parts of the country improvements in organization and methods of instruction in mathematics have been made these last years. All persons who are, or may become, actively interested in this movement of reform should in some way unite themselves, in order that the plans and the experience, whether of success or failure, of one may be immediately made available in the guidance of his colleagues.

I may refer to the centers of activity with which I am acquainted. Miss Edith Long, in charge of the Department of Mathematics in the Lincoln (Neb.) High School, reports upon the experience of several years in the correlation of algebra, geometry and physics, in the October, 1902, number of the *Educational Review*. In the Lewis Institute of Chicago, Professor P. B. Woodworth, of the Department of Electrical Engineering, has organized courses in engineering principles and electrical engineering in which are developed the fundamentals of practical mathematics. The general question came up at the first meeting* (Chicago, November, 1902) of the

*Subsequent to the meeting of organization in the spring of 1902. Mr. Chas. H. Smith of the Hyde Park High School, Chicago, is president of the Association. Reports of the meetings are given in *School Science* (Ravenswood, Chicago.)

Central Association of Physics Teachers, and it is to be expected that this association will enlarge its functions in such a way as to include teachers of mathematics and of all sciences, and that the question will be considered in its various bearings by the enlarged association. At this meeting informal reports were made from the Bradley Polytechnic Institute of Peoria, the Armour Institute of Technology of Chicago, and the University of Chicago. The question is evoking much interest in the neighborhood of Chicago.

I might explain how I came to be attracted to this question of pedagogy of elementary mathematics. I wish, however, merely to express my gratitude to many mathematical and scientific friends, in particular, to my Chicago colleagues, Mr. A. C. Lunn and Professor C. R. Mann, for their cooperation with me in the consideration of these matters, and further to express the hope that we may secure the active cooperation of many colleagues in the domains of science and of administration, so that the first carefully chosen steps of a really important advance movement may be taken in the near future.

I close by repeating the questions which have been engaging our attention this afternoon.

In the development of the individual in his relations to the world, there is no initial separation of science into constituent parts, while there is ultimately a branching into the many distinct sciences. The troublesome problem of the closer relation of pure mathematics to its applications: can it not be solved by indirection, in that through the whole course of elementary mathematics, including the introduction to the calculus, there be recognized in the organization of the curriculum no distinction between the various branches of pure mathematics, and likewise no distinction between pure mathematics and its principal applications? Further, from the standpoint of pure mathematics: will not the twentieth century find it possible to give to young students during their impressionable years, in thoroughly concrete and captivating form, the wonderful new notions of the seventeenth century?

By way of suggestion these questions have been answered

in the affirmative, on condition that there be established a thoroughgoing laboratory system of instruction in primary schools, secondary schools and junior colleges—a laboratory system involving a synthesis and development of the best pedagogic methods at present in use in mathematics and the physical sciences.

SUGGESTIONS FOR THE SOLUTION OF AN IMPORTANT PROBLEM THAT HAS ARISEN IN THE LAST QUARTER OF A CENTURY*

By Raleigh Schorling

INTRODUCTION

It is hoped that the general theme of this program will prove interesting and profitable but it is not likely that we can accurately judge the significance of the various movements that have grown out of the last twenty-five years. We are probably too close to the picture to see it in its true perspective. It is, nevertheless, interesting to speculate as to what the future student of history will write about our times.

Progress comes through people. I believe that no true appraisal of the last twenty-five years can be written without associating the contributions with the personalities responsible for them.

We can only guess what will be the list of names which will persist. But it is fairly safe to guess that the name of Professor E. H. Moore will be first. Not quite twenty-five years ago he delivered the address, reprinted in this volume, which profoundly stimulated progressive thinking on the teaching of high school mathematics. I often find it most interesting to read this address and to note the many issues clearly sounded there which still challenge our best efforts. We have not yet achieved the goals set up for us in that brilliant vision. May I mention just a few valuable things that seem to me to have come to us from Professor

*At the annual meeting held in Cincinnati, Ohio, (1925) the National Council of Mathematics Teachers passed a motion directing the president to deliver an address at the next meeting.

Moore's address. Directly it stimulated the teaching of intuitive, or experimental geometry in this country. It started a vigorous reaction to the formalism found in the algebra of Wentworth. It placed emphasis upon the interrelations which a thoughtful teacher can build between the topics chosen from various branches, arithmetic, algebra, geometry, and trigonometry.

At this time it seems reasonably safe to add a few more names. To Professor George Myers belongs the credit for initiating and directing the pioneer effort to formulate the inter-relations of the various branches for the use of the classroom teachers. The English writer, Nunn, is little known by American teachers but I predict his place also as being secure. In his writings he exhibits a marvelous insight as to how children master the introductory concepts of mathematics. About ten years ago American texts for children began to show the influence of Nunn. More recently he has affected the pedagogical writings of some American authors. Nor can there be any question about John Dewey. He has probably changed the teaching of mathematics and all the other school subjects more than any other living man. To him we owe such guiding principles as we now accept. It was Dewey who started us teaching children instead of school subjects, and it was Dewey who said "Education is not preparation for life but is Life." To the psychological group we are indebted for the systematization and formulation of certain general laws of learning so as to be genuinely helpful to the teachers of special subjects. As a result writers have keener insight into the manner in which children learn and are able to state problems closer to the pupil's point of view. This contribution is valuable in this day when our secondary schools are crowded with pupils who have little background and experience and less ability for mathematical training. The contribution of J. W. Young to the work of the National Committee on Mathematical Requirements will surely gain him a place on the list of outstanding contributors of this quarter of a century. As a member of that committee I testify to the value of his services.

His gift lay, so it seemed to me, in his rare ability to take home with him the conflicting views of others and in a few weeks return these formulated so skillfully as to be acceptable to all,—and that without compromise or sacrifice of the fundamental principles. Often people differ only in terminology rather than in purposes. It was most fortunate for the teachers of mathematics in our secondary schools that a man of Professor Young's standing and ability was willing to sacrice several years of his life in devotion to our problems. To David Eugene Smith we are indebted for creating a mine of information in the history of mathematics that surely someone will sooner or later make available to our girls and boys. There must be many who hope as I do that Professor Smith will make this his own crowning achievement. It would be futile to extend this list of distinguished contributors to whatever progress has been achieved in these twenty-five years. As I have already implied, no one should take the preceding predictions too seriously as this brief list is given merely by way of illustration.

By action taken at the last annual meeting at Cincinnati, I, as retiring president, am obligated to deliver this address. Since no subject was assigned, I chose the general theme of the yearbook with an emphasis on what seems to be a crucial problem. This problem has arisen from the changing conditions which have crowded over three million girls and boys, one in every three of high school age, into the schools.

The Problem: How to Improve the Quality of Scholarship.

The most significant factor in the present situation I believe to be our realization of the very low mastery achieved in the secondary schools,—in both the junior and senior high school grades. For years we have heard sharp criticism coming from the business world, indictments by college teachers of mathematics, and resentment expressed by administrators at the high number of pupil failures but it remained for a by-product of the testing movement really to open our eyes. It is now possible to turn to many elements drawn from

numerous tests which have been given to thousands of children in many communities. For evidence on this point consider first the following table drawn from a more extensive one recently completed by the writer. The complete Inventory* Test consisting of 125 elements was submitted to many children. The table given includes some of the elements in the Inventory Test and also the percentage of correct responses for 3,260 beginning seventh grade pupils. The table should be interpreted as follows:

Of the 3,260 pupils tested, approximately 85.3 per cent responded correctly to the situation "Write 3 per cent as a decimal." Looking at task number 84, we see less than half the pupils (more accurately 47.5%) were able to write 0.125 as a common fraction.

MASTERY IN ARITHMETIC

	Test Elements		Percentage of Correct Responses
70.	Write 3% as a decimal. Ans.	70.	85.3
71.	Write 6½% as a decimal. Ans.	71.	64.4
72.	Write .4 as per cent. Ans.	72.	35.8
73.	Write .025 as per cent. Ans.	73.	37.2
74.	Write ⅔ as a decimal. Ans.	74.	66.6
75.	Write ⅛ as a decimal. Ans.	75.	71.4
76.	Write ¾ as a decimal. Ans.	76.	74.8
77.	Write ½ as per cent. Ans.	77.	73.5
78.	Write ¼ as per cent. Ans.	78.	74.1
79.	Write ⅛ as per cent. Ans.	79.	70.1
80.	Write ⅓ as a decimal. Ans.	80.	62.9
81.	Write $\frac{1}{16}$ as a decimal. Ans.	81.	55.4
82.	Write $\frac{1}{12}$ as a decimal. Ans.	82.	35.6
83.	Write .25 as a common fraction. Ans.	83.	81.3
84.	Write .125 as a common fraction. Ans.	84.	47.5
85.	Write .06¼ as a common fraction. Ans.	85.	27.2
86.	What is 3% of 200? Ans.	86.	59.7
87.	The ratio of 1 to 2 is equal to the ratio of 5 to	87.	33.0
88.	The interest on $200 for 3 years at 4% is $	88.	60.8
89.	Fred Stone found that 5 per cent of 200 is 10. What number is the percentage? Ans.	89.	19.3
90.	Fred Stone found that 5 per cent of 200 is 10. What number is the base? Ans.	90.	48.0
91.	Theodore Roosevelt was born October 27, 1858, and died January 6, 1919. His age was .....years, ...months, ...days.	91.	29.7

*For a more extensive discussion see "A Tentative List of Objectives in the Teaching of Junior High School Mathematics, with Investigations for the Determining of Their Validity." George Wahr, Publisher, Ann Arbor, Mich.

The next table may give the reader a clearer picture of what was included in our Inventory Test for there the tasks are chosen from different sections of the test. As in the preceding table the figures in the column at the right indicate the percentage of correct responses from 3260 pupils.

Note that in task number 76 less than 75 per cent were able to write 3/4 as a decimal. Only about two out of every five pupils were able to respond correctly to the question

OTHER ITEMS OF THE INVENTORY TEST*

51. Look at this $5 \times 4 = 20$. Which number is called the product?..(78.4)
42. Write in figures: Fifty-nine and three hundredths.....(76.1)
5. One gallon equals how many pints?.....(75.6)
76. Write $\frac{3}{4}$ as a decimal.....(74.8)
77. Write $\frac{1}{2}$ as per cent.....(73.5)
56. Fifty per cent of a number is the same as what part of that number?.....(71.4)
74. Write $\frac{1}{5}$ as a decimal.....(66.6)
88. The interest on $200 for 3 years at 4% is $.....(60.8)
7. One square foot equals how many square inches?.....(60.3)
96. Draw a figure to show that you know the meaning of the word rectangle.....(59.8)
86. What is 3% of 200?.....(59.7)
18. April has.....days; November has.....days; July has .....days?.....(55.0)
65. What must you do to find $\frac{2}{3}$ of $\frac{5}{7}$? Do you add? If not, what must you do?.....(54.9)
58. % means.....(51.4)
32. If you have the product of two numbers, how can you find the other number?.....(44.5)
117. Be able to identify a check when a check is shown.....(43.4)
99. Draw a figure to show the meaning of "right angle."....(43.0)
36. What is the average of 4, 6, 8 and 10?.....(42.7)
29. Moving the decimal point one place to the left.....the number by 10.....(42.1)
40. Does 4896 ÷ 10 equal 4.896, or 48.96, or 489.6, or 4896? Draw a circle around the right number.....(35.5)

"What is the average of 4, 6, 8, and 10?" The preceding table is of special interest because 146 experienced teachers, when asked to check the items of the complete test which they felt

*For diagrams and illustrations see copy of the Schorling-Clark-Rugg Inventory Test, Gazette Press, Yonkers, New York.

sure pupils in the beginning of the seventh grade should know with nearly 100 per cent accuracy, chose among other items those appearing in that table.

The elements of the Inventory Test were chosen on the following bases: (1) Appearance in courses of study. (2) The judgments of twenty-nine writers of arithmetics. Moreover, it is important to remember that this test was practically an untimed test.

A few of the questions have subtleties, or were stated in unfamiliar forms, but on the whole the tasks were simple for seventh grade children. Week after week and year after year some of these tasks are given systematic drill to fix certain skills. It is said society spends from one-fourth to one-fifth of the money devoted to the intermediate grades in the teaching of arithmetic. So we may ask whether anything learned to the point where less than sixty per cent of the children respond accurately while the material is fresh in mind is worth very much to society.

THE SUMMARY TABLE

The Table which follows represents a summary of the results of the Inventory Test. The table should be interpreted as follows: There were only eight tasks in the Inventory Test that as many as 90% of the pupils tested could do. There were twelve exercises that received from 80% to 90% correct responses. There were five tasks so difficult that less than 10% of the pupils succeeded in each.

SUMMARY TABLE

Per cent	No. of Elements
90–100	8
80– 90	12
70– 80	14
60– 70	10
50– 60	14
40– 50	22
30– 40	14
20– 30	14
10– 20	12
0– 10	5
	125

Are We Satisfied with Low Standards?

Many European visitors to our schools, though they may be enthusiastic about many excellent characteristics, are vigorous in their indictment expressed in such phrases as: "You foster half learning"; "You are satisfied with low standards"; "You do not fix habits"; and, "There is no question that you can ask your children that any considerable number of them will know." It is not clear what is meant by "half learning." Perhaps it means that half of the children learn all the things that we try to teach them, or that half of the things are learned by all the children. The table suggests that neither happens. A few of the items in the test are learned by nearly all the children, a few elements are mastered by only a few children, and a large number of tasks (the mode) falls between 40% to 50% mastery. In fact, the Inventory Test resulted in something very close to a normal distribution. By way of summary, it may be stated that the total number of tasks getting 50% correct responses is 58 out of 125—less than half the test. If by half learning is meant that each and every one of half the number of outcomes is mastered by less than half the pupils, there is evidence that the criticism is none too harsh. In fact, we do not achieve anything like half learning in this sense. We are driven to admit that pupils lack mastery of the essentials of arithmetic.

Does the Same Condition of Low Achievement Hold in Ninth Grade Algebra?

On this point we can speak quite definitely for Thorndike has reported a similar study on the strength of algebraic connections.* He reports on twenty-eight tasks. His table includes records secured from ten schools though the number of pupils tested is not given. The table on the next page shows the results secured on six of the tasks by the four lowest schools. The percentages given are those for correct responses.

It is true that some of the required tasks might be criticized as including subtleties that are not ordinarily touched

*"The Psychology of Algebra," Chapter XII.

by drill. Thus the arrangement in Exercise 4 adds to the difficulty. A direction or a challenge of some sort could well have been added to Exercise 9. Teachers do not give much practice on such equations as are represented in Exercise 26, and the statement for Exercise 27 could probably be improved. Indeed, it is likely that Thorndike used these subtleties to demonstrate to teachers the need for more varied drill. On

MASTERY OF ALGEBRA

Modified from Thorndike's Table	Percentage of Correct Responses			
(4) From $5a-b-2c$ subtract $3c-3a$	49	68	41	35
(7) $7d \times 2de^2$	74	70	72	74
(9) $8-5\ (d+2)$	71	39	20	37
(17) If $a=2$ and $b=3$ what does $5a-2ab$ equal?	68	56	38	49
(26) $15=7W-4$ what does W equal?	67	45	42	51
(27) $\frac{V}{4}=a-2$ what does V equal?	51	38	26	26

the whole, it must be admitted that these exercises are very simple when contrasted with the complex manipulations required by a standard ninth grade algebra course. Thorndike concludes, "It does not seem an exaggeration to say that on the whole these students of algebra had mastery of nothing whatsoever. There was literally nothing in the test that they could do with anything like 100% efficiency." This is a severe statement of the results secured from high school people who are undertaking their second year of algebra.

The results secured by Hotz's test furnish additional evidence. The following table consists of elements selected from the table which appears in Thorndike. Here also the table has been modified to include percentage of correct responses. The pupils tested had studied algebra for nine months.

The particular elements selected are very much easier than the exercises ordinarily expected of ninth grade algebra stu-

dents at the end of the year. Here, as in the Thorndike results, we find a very low degree of mastery. The situation re-

RESULTS ON THE HOTZ TEST

	Percentage of Correct Responses		Percentage of Correct Responses
$\frac{3c}{4}-\frac{3c}{8}=$	64	In the formula RM = EL find the value of M	51
$(-3xy^3)^4$	62		
$\frac{y}{3}=\frac{5}{2}+\frac{y}{4}$	70	The area of a circle = πr^2 in which r = radius of the circle and π = 3 1/7. Find the area of a circle whose radius is 7 ft.	48
$\frac{2x}{3}=\frac{5}{8}$	80		

vealed by the Inventory Test for arithmetic is probably worse in ninth grade algebra, for pupils get little practice on the algebraic skills in out-of-school activities. The results of Thorndike, Hotz, Childs, and Monroe, all support the belief. By way of summary, we may quote Thorndike, "Complicate the situation slightly and the pupils fail."

There is a wealth of evidence from surveys that could be used at this point. Courtis found that the answers of grade seven on his test 7 were about 57% accurate. Support also could be drawn from the measurements made by students in psychology who have used arithmetic exercises as tests of fatigue, of individual differences, etc. For the summary of the evidence the reader may turn to Thorndike, "The Psychology of Arithmetic," page 103. Consider finally the results recently secured on the Woody-McCall Form 3. This test was widely administered in the state of Michigan.* The results in the following table are for May, 1924, and include the results from forty cities. The cities were divided into three pupil groups. Group I includes cities having 5,000 or fewer inhabitants; Group II includes cities having 5,001 to 10,000 inhabitants; and Group III includes cities having over 10,000 inhabitants. On the first problem: 2+4=, 114 pupils

*The test is not included. It is assumed that anyone reading this study up to this point is familiar with the Woody-McCall Test.

out of 1626 pupils failed. The same number of pupils could not respond correctly to 6×2=. Nearly 200 pupils failed to answer 5×1=. Let us consider next a few questions only a little more complicated. On test 17, namely, 31—1=, nearly

RESULTS ON THE WOODY-McCALL FORM III*

Number of Problem	Grade VII Group I	II	III	All	Number of Problem	Grade VII Group I	II	III	All
1	91	84	98	93	20	75	88	85	83
2	90	85	98	93	21	61	70	71	68
3	91	85	99	94	22	76	82	76	77
4	92	89	98	95	23	57	70	68	66
5	83	92	91	89	24	52	49	56	53
6	84	94	92	91	25	76	85	82	81
7	78	83	85	83	26	57	69	66	65
8	86	89	89	88	27	47	63	63	59
9	90	83	97	92	28	37	49	43	43
10	87	92	93	91	29	51	58	62	58
11	86	96	93	92	30	18	23	34	28
12	88	97	95	94	31	20	22	25	23
13	82	85	88	86	32	21	34	34	31
14	83	93	89	88	33	17	28	14	18
15	84	92	88	88	34	9	12	14	12
16	75	86	86	83	35	14	19	24	20
17	69	77	70	71					
18	82	94	84	86					
19	68	80	79	77					
					Number of Pupils	406	369	851	1626

30 out of every 100, or 472 out of 1626 pupils failed to respond correctly. More than this number failed on test 26: 5939×85=. On exercise 31, namely, 7/8÷6=, we find nearly 3 out of every 4, or a total of 1252 pupils out of 1626, failing to write the correct answer.

These results are recent and they represent a wide sample of the state. Woody believes that these low results are in no way due to insufficiency of time,—most pupils finish in ten minutes though the test allows twenty. Furthermore, he believes that these results are slightly higher than those generally secured from the many cities contributing to the development of the Woody standards.

*This table was furnished to the writer by Professor Clifford Woody.

This concludes our evidence. Since our study was first undertaken Thorndike has reported much evidence on this issue. The results of the two studies are in entire agreement, as may be seen by a reading of Chapter V in Thorndike's "The Psychology of Arithmetic." The general trend of his argument appears in the following: "It is clear that numerical work as inaccurate as this has little or no commercial or industrial value. If clerks got only six answers out of ten right, as in the Courtis tests, one would need to have at least four clerks make each computation and would even then have to check many of their discrepencies by the work of still other clerks, if he wanted his accounts to show less than one error per hundred accounting units of the Courtis size. It is also clear that the habits of . . . absolute accuracy, and satisfaction in truth, a result which arithmetic is supposed to further, must be largely mythical in pupils who get right answers from only three to nine times out of ten!"

The Passing Mark in Secondary Schools.

In a recent conference it was developed that the school officials commonly report the passing mark as being, 70, 75, 80, or even 85 per cent. In the light of the evidence here submitted it must be clear that so high a passing mark is a *myth.* A passing mark of 75 or even one so low as 60 per cent could not possibly be supported by test results now in existence. It is interesting to speculate what kind of a test one would need to give in order that a considerable number of pupils would be able to respond correctly to 75 per cent of the test elements. The tasks would need to be very, very simple and of the types which receive considerable drill in the home, in the corner drug store, in the grocery store, on the play ground and in numerous other life situations. Returning once more to our Inventory Test there are only eight items which receive 90% correct responses or better, and these items are the simple tasks: "One dozen equals how many things? One pound equals how many ounces? One hour equals how many minutes? One minute equals how many seconds? If you have two numbers,

how can you find their sum? What is 12÷3? and, Make drawings showing that you know the meaning of the words 'circle' and 'square'." At least four of these items are learned by a considerable number of children before they enter the first grade.

It does not seem that 80% mastery is an unreasonably high standard to expect on items that the school has definitely included in its program, but if we are willing to accept 80% as our standard, we get only twelve additional tasks done at this level. These are: "One yard equals how many inches? One bushel equals how many pecks? If you have two numbers, how can you find their product? If you have two numbers, how can you find their difference? If you know the cost of a certain number of pencils, how can you find the cost of one of them? What is the difference between eight and five? Look at this: $16 \div 8 = 2$, which number is called the quotient? Look at this: $9 + 7 = 16$. Which number is called the sum? Write 3% as a decimal. Write 0.25 as a common fraction. Make a drawing to show you know the meaning of the word, 'triangle'." The reader will agree that all these are simple tasks. Many children learn at least three of these before entering the first grade and it is possible that a considerable number learn all but two of these before completing the third grade.

We have chosen our evidence from one of the most definite of school subjects and we have selected tasks that are commonly found as objectives in courses of study. What do we achieve in the less well defined subjects if our attainment is so low in a well standardized field? It is not likely that questions in the social studies excepting possibly one concerning the discovery of America, could be asked of three thousand beginning seventh grade pupils with anything but low returns.

It is not clear how teachers arrive at the high passing mark. Possibly we give a mark from 80 to 100 for "oral recitations," another equally high mark for general cooperative spirit or discipline, then weight these marks so that out of this computation a comfortable passing mark of 80 per cent

emerges. Instead of a passing mark of 75% or 80%, it is the writer's conviction that the real passing mark of our American secondary schools at the present is somewhere between 40 per cent and 50 per cent.

A Need for Action.

The lesson to be learned from these figures is that there is no time to be lost. The crucial question before us, the teachers in the secondary schools, is *how to improve the quality of scholarship.* We can easily explain the situation by saying that the population of the secondary schools has increased rapidly. Undoubtedly we have not been able to meet the situation with teachers adequately prepared. We may say also that, since we are getting a larger sampling in our high schools from society, the chances are that the general level of ability has been sharply lowered. We may add, moreover, that the pupils are no longer of the same type as were those in our high schools in the nineties. Many come from "first generation homes." They do not even speak our language. Every school subject now has unusual difficulties with the vocabulary of the subject. We live in days in which concentration on purely scholastic matters is infinitely more difficult. Throbbing life all about young people pulls the attention away from scholastic activities. Although all this may be true it yet remains that we, the class room teachers, must take prompt action before the public generally recognizes just what is happening. Confusion may result if society concludes that it is not getting a just return for its huge investment in secondary schools.

Even if we should undertake remedial measures now there is no guarantee that the changes proposed will be given enough time,—a fair chance to prove their worth. For example, the writer knows professors of mathematics in college departments who are bitterly critical of the mastery shown by the product that comes to them from the high schools. They promptly put the blame upon general mathematics, upon correlated mathematics, or upon the program of the National Committee. The fact is that of all of the students who now

come to colleges and universities very few have studied general mathematics or correlated mathematics, and certainly the students could not yet have been harmed by the program of the National Committee on Mathematical Requirements. These men have listened to the discussion of the newer things, or they have read about them in educational journals, and they conclude that the very poor material with which they deal is directly due to these recent proposals. Whereas the clear fact is that this material is almost altogether the product of "good old fashioned materials and methods." So far as the writer knows there is only one college or university that may be considered an exception, and the troubles of that institution seem to be due to accepting credits in "scrambled" plane geometry. If high grade men in the university are impatient in giving the newer courses time to prove their worth,—a real chance, how patient may we expect the general public to be with our shortcomings?

A Program for Improvement.

At least six measures can safely be taken to improve the quality of our scholarship. (1) We need to formulate a basic philosophy for the guidance of our work. We dare no longer drift on an uncharted sea. We need guiding principles to help us in the selection of materials, in the choice of methods, and in the placement of emphases. (2) We need to formulate in a specific way the outcomes that we desire and the degree of mastery for which we strive. If we want our pupils to know a certain principle or a specific formula, the superintendent, the principal, the teacher and the pupils must all be aware of that objective. It is futile to expect to secure results in mathematics by the "shotgun" method. It is clear from our test results that the mastery of any task, unless so simple that it needs no formal instruction, is not a *by-product.* We must strive for it definitely. (3) In selecting our specific objectives we need to place emphasis on those elements of the curriculum for which a positive case can now be made by the use of one or more *objective* studies of the use of mathematics. Our work

in curriculum building at best will be rough enough. There is no excuse for neglecting the elements that do have support on the basis of social need. A course of study based on the summary of twenty-nine objective studies which have more or less bearing on what should be taught in junior high school grades will be submitted. (4) We need to employ certain well-accepted principles from the psychology of drill. Obviously the ordinary algebra is weak in its application of the psychology of drill, this in spite of the fact that the field of educational psychology seems to have made its most valuable contribution of recent years at this point. A list of principles for drill purposes which the writer has found helpful in his own daily class room work will presently be presented. (5) We need to construct our teaching materials under precisely controlled and tested conditions. Most text books are still being written by persons quite removed from the activities of the pupils for whom they are intended, published without experimental use, and sold for a number of years, until an avalanche of criticism compels revision. The revision results in changes, the need for which would have been obvious had the text been tried out in a limited, carefully controlled manner. It is astonishing to see that some writers are only now discovering the need for timed practice exercises in algebra, apparently ignorant of the fact that such materials growing out of detailed studies of children's responses have been available for years in a few progresive books. (6) We need a basic, introductory course (general mathematics) in grades seven and eight and for certain groups in the ninth grade.

The preceding remedial measures will now be discussed in some detail.

I. THE GUIDING PRINCIPLES

A tentative list of guiding principles have been proposed elsewhere* and need not be repeated here. However, the read-

*See A Tentative List of Objectives In the Teaching of Junior High School Mathematics—With Investigations for the Determining of Their Validity. George Wahr, Publisher, Ann Arbor, Mich.

er should have these in mind as we proceed to the next step.

II. SPECIFIC OBJECTIVES

The next remedial measure suggested is the formulation of a list of *specific objectives.*

There was nothing indefinite about the work of the algebra teacher in the nineties. But there is danger that the newer education in spite of its many good points will drift into the vague, as may be seen by a study of content of the first seven series of books written for the junior high school. The material included in these books is suggested in the following table.

TEXT	SEVENTH		EIGHTH		NINTH	
	1st half year	2nd half year	1st half year	2nd half year	1st half year	2nd half year
A.	Arith.	Int. Geom.	Algebra	Arith.	Algebra and Trig.	Plane Geometry
B.	Arith.	Geometry	Arith.	General Math.	Algebra	Algebra
C.	General Math.	General Math.	General Math.	Arith.	Algebra	Algebra and Trig.
D.	Arith.	Arith.	General Math.	General Math.	Algebra	Plane Geometry
E.	General Math.	General Math.	Algebra	Algebra	Arith.	Arith.
F.	Arith.	Arith.	Int. Geom.	General Math.	General Math.	Gen. Math. and Trig.
G.	General Math.	General Math.	General Math.	General Math.	Algebra	Plane Geometry

This table reveals an astonishing degree of variation. It would seem difficult for a parent to predict what large field of mathematics his child will be expected to study. Election in mathematics might mean arithmetic, geometry, algebra, or general mathematics. This is even true of the last half of the ninth grade. Three of the books offer plane geometry, one, arithmetic only, one, general mathematics including a unit of trigonometry, and two offer algebra, each including a unit of trigonometry. Is there any wonder that school executives, supervisors, committees working on courses of study, to say

nothing of the beginning teachers, are confused by the wholly different offerings presented? We have here evidence of distinctly different hypotheses concerning the organization of materials for these grades.

Not only do we need a list which will reveal the specific concepts we wish to develop, the special skills we desire to fix, the definite items of information we want to impart, the particular attitudes we expect our pupils to have, but we need also to be able to give at least one reasonably good answer for our choice of each element. It is a fair question for the pupil, parent, or tax payer to ask, "Why do you teach these particular things?" To bring this about for his own teaching, the writer has recently completed a study which eventuates in a long list of specific objectives. For three hundred five elements he is able to give at least three answers to the question "Why teach this item?" For one hundred forty-nine other items he is able to give at least one or more. The procedure was as follows:

I. The formulation of a brief list of guiding principles implying a philosophy of secondary education.

II. The selection of specific objectives on the basis of the following five criteria:

   a. A summary of the elements for which some kind of a positive case can now be made by employing one or more objective studies dealing with the social use of mathematics.

   b. Practice as determined by an inventory of a selected set of courses of studies:

      1. For seventh and eighth grades: Guiler's Thesis.
      2. For ninth grade: A study by the author.

   c. The outline of topics as given on pages 21 to 27 in the report of the National Committee on Mathematical Requirements.

   d. An inventory of the contents of the first seven series of mathematics texts written for the junior high school grades.

e. The decisions of a highly selected jury of five educators especially interested in the junior high school, and five leaders in the teaching of high school mathematics.

III. The refinement of the elements selected and their grade placement by means of an extensive classroom trial under test and report conditions.

Obviously there is not space here to present the extensive tables of this study.*

III. A COURSE OF STUDY BASED ON OBJECTIVE STUDIES

The third remedial measure suggested is to examine carefully our social needs as we build the list of specific objectives. Presumably no intelligent person would nowadays construct a course of study without asking such questions as

What mathematics do people; e.g., the grocery clerks, the pharmacists, foremen in shops, etc., need in common life?

What mathematics do high school girls and boys need as they go about their courses in industrial arts, mechanical drawing, fine arts, general science, physics, chemistry, and the like?

What mathematics do girls and boys need to interpret the newspapers and magazines that they are supposed to read?

What mathematics do the people need who go to college and study the mathematical sciences and the social sciences?

But the probability is that we shall have many lists which will ignore completely all available studies of a scientific character. Already we have two extensive lists that represent nothing more than *opinion,*—their author's and the assent of a group of teachers (as if we did not have numerous courses of study made in precisely this fashion though fortunately lacking the educational "patter").

*For the complete list see "A Tentative List of Objectives in the Teaching of Junior High School Mathematics, with Investigations for the Determining of Their Validity." George Wahr, Publisher, Ann Arbor, Mich.

There are about thirty studies available which have more or less bearing on the selection of materials in mathematics for our secondary schools. Most of these have a greater significance for the mathematics of the junior high school than for that of the senior high school. The writer has recently summarized twenty-nine studies grouped as follows:

(1) Analyses of pupil activities,—one study, Chase.

(2) Studies of the uses of Mathematics in general reading, six studies, Adams, Schorling (4), and Thorndike.

(3) Studies of the social and business activities of adults, nine studies, Charters, Chase, Mitchell (4), Wilson, Wise, and Woody.

(4) Investigations of the academic uses of mathematics by high school and college students in subjects other than mathematics; six studies, Callaway, Rugg and Clark (2), The National Committee, Thorndike, and Williams.

(5) Questionnaire studies, seven,—those by Jessup and Coffman, Wilson, Camerer, Moore (3), and Woody.

We may now ask the question "What mathematics would we teach if we taught only those elements for which some kind of a positive case can be made?"

In the following table these items are listed, together with names in parentheses which indicate the references where possibly the strongest evidence may be found:

A Course of Study Based on Objective Studies.

1. High skill in the ability to estimate answers for checking results secured by automatic mechanical devices in the business world. (Woody)
2. Reading large numbers, probably to billions. (Adams)
3. Knowledge of common units of foreign money—the English pound, the French franc, and the like. (Adams)
4. Familiarity with a large variety of denominate numbers. (Adams, Schorling, Williams, Woody)
5. Familiarity with a long list of units of measure, probably more extensive even than found in our school texts. (Adams, Schorling)
6. The rational use of significant figures. (Nat. Com.*)

*Hereafter Nat. Com. will refer to the National Committee on Mathematical Requirements.

7. Numerical computation with approximate data. (Nat. Com.)
8. The meaning and use of the elementary concepts of statistics. (Nat. Com.)
9. Computation with short-cut methods. (Nat. Com.)
10. Taking square root of arithmetic numbers. (Thorndike)
11. Understanding the use of mechanical devices—weighing, measuring, computing, and the like. (Woody)
12. Skill in the extensive use of tables—bonds, interest, percentage solution, screen, discount, printers, and the like. (Woody)
13. Familiarity with the metric system. (Nat. Com., Schorling, Woody)
14. Graphical representation of statistical data: (Nat. Com.)
    (a) Bar, line, and circle graphs to represent ratio and dependence. (Nat. Com.)
    (b) Reading graphs showing relation of one variable to another. (Thorndike)
    (c) Critical faculty in reading and in evaluating graphs. (Thorndike)
    (d) Presentation of laws by mathematical graphs. (Nat. Com., Thorndike)
15. The reading of geometric figures by means of letters designating points. (Gallaway)
16. Familiarity with numerous geometric terms, a list probably far more extensive than found in the mensuration of arithmetic. (Gallaway, Schorling)
17. Understanding of an extensive list of geometric concepts, most of which can probably be taught to children by simple intuitive and experimental methods. (Schorling)
18. The use of simple geometric constructions: (Gallaway)
    (a) Drawing a line segment of a given length.
    (b) Drawing a line parallel to a given line.
    (c) Drawing a line perpendicular to a given line.
    (d) Drawing a circle or an arc of a circle having given the center and the radius.
19. Command of the fundamentals when applied to integers:
    (a) For addition the following suggestions seem important:
        1. Addends of four digits. (Charters)
        2. Six addends and not more than five places. (Woody)
        3. Five addends with five digits. (Moore)
    (b) For subtraction not more than four places in the minuend. (Woody)
    (c) For multiplication we find:
        1. The multiplicand five digits, and the multiplier four digits. (Moore)
        2. The multiplicand twelve or less with multiplier three places or less. (Charters, Woody)
    (d) Division:
        1. The dividend five digits, and the divisor four digits. (Moore)

2. Three place numbers in dividend are most common. (Woody)

20. High skill in applying the fundamentals to common fractions with denominators, 2, 3, 4, 5, 6, 8, 10, 12, 16, 32. This list, brief as it is, allows for a "safety factor"—it is considerably longer than the one suggested by Thorndike and much more extensive than that of the National Committee on Mathematical Requirements. (Schorling, Wilson, Wise)
21. To guide us in applying fundamentals to mixed numbers the following are significant:
    (a) Mixed decimal by mixed decimal (16 cases). (Williams)
    (b) Multiplying two fractions. (Gallaway)
    (c) Multiplication by a fractional, integral, or mixed number of dozens. (Mitchell)
    (d) Integer by decimal. (Williams)
    (e) Mixed decimal divided by mixed decimal. (Williams)
    (f) Addition of any number of twelfths to integers and mixed numbers. (Mitchell)
    (g) Reducing mixed numbers to improper fractions. (Gallaway)
    (h) Almost any decimal up to four places. (Williams)
    (i) Goods frequently marked in decimal system. (Woody)
    (j) Multiplying five digit number with three decimal places by five digit number with two decimal places. (Williams)
22. Ability to use simple business forms: family expense accounts, checks, bills, notes, deposit slip. (Camerer, Moore)
23. Knowledge of the use of discount, interest, opening bank account, etc. (Camerer, Moore)
24. Knowledge of saving and loaning money, mortgages, building and loan, simple accounts, thrift bonds and real estate investments, taxes, and levies. (Wilson)
25. Acquaintance with insurance, taxation, and thrift. (Jessup and Coffman)
26. Ability to interpret percentage in a great variety of ways. (Adams)
27. Ability to find per cent of a given number. (Williams)
28. Ability to interpret ratio in a great variety of ways. (Adams, Nat. Com.)
29. Ability to use proportion. (Nat. Com., Williams)
30. Ability to use the simple machinery of algebra:
    (a) Use axioms. (Nat. Com., Thordike)
    (b) Read symbols.
    (c) Know the meaning of the omission of the times sign.
    (d) Know the meaning of the omission of one as a coefficient.
    (e) "Letting — = the number of —," (Thorndike)
31. Ability to translate a verbal statement into symbolic. (Gallaway)
32. Ability to deal with a single parenthesis. (Thorndike)

33. Ability to deal with a simple fractional exponent and transformation of the type $\sqrt[3]{a} = a^{1/3}$. (Thorndike)
34. Knowledge of positive integral exponents. (Thorndike)
35. The ability to interpret negative numbers. (Nat. Com., Thorndike)
36. The evaluation of simple algebraic expressions. (Nat. Com., Rugg and Clark)
37. The ability to apply fundamentals to simple algebraic terms, as for example, multiplication of two binomials. (Rugg and Clark)
38. The ability to apply fundamentals to simple fractions. (Rugg and Clark)
39. The ability to clear a proportion of fractions—great skill in the use of the proportion form of equation. (Thorndike)*
40. The ability to form correct proportions. (Rugg and Clark, Thorndike)
41. The ability to do the simplest cases of factoring. (Rugg and Clark, Thorndike)
42. Understanding of the linear function, $Y = mx + b$. (Nat. Com.)
43. Ability to use equations of the first degree and one unknown. (Rugg and Clark)
44. Ability to use simultaneous linear equations in two unknowns. (Nat. Com., Williams)
45. Ability to use fractional equations. (Williams)
46. The ability to use quadratic equations in one unknown. (Nat. Com.)
47. The ability to obtain L. C. M. by inspection in the case of a fractional equation with simple numerical denominators. (Rugg and Clark)
48. Ability in the use of the formula:
    (a) Ability to read simple formulas.
    (b) Ability to solve simple formulas.
    (c) Ability to evaluate simple formulas.
    (d) Ability to construct simple formulas.
    (e) Ability to change the subject of a formula.
    (f) Ability to substitute in a formula. (Nat. Com., Thorndike)
49. Some skill in the solution of verbal problems. Those dealing with:
    (a) Proportion, such as mixtures. (Thorndike)
    (b) Buying and selling goods. (Wilson)
    (c) Labor and wages. (Wilson)
    (d) Interest. (Wilson)
    (e) Rent. (Wilson)
    (f) Insurance. (Wilson)

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*Contradicts other evidence but the writer is convinced that Thorndike is correct.

(g) Measurement of capacity. (Wilson)
(h) Estimates of building materials. (Wilson)
50. Ability to use logarithms in computation. (Nat. Com.)
51. Ability to use sine, cosine, and tangent in the solution of simple problems of the right triangle. (Nat. Com., Rugg and Clark)

The Inadequacy of Any Single Objective Study.

It ought to be unnecessary to point out that any one of the so-called scientific studies now available cannot serve as the basis for the selection of subject matter. But it is not infrequent that much publicity is given to a single study holding out the hope that our curriculum problems are being solved by it alone. In the summer of 1924 the newspapers of our Nation's capitol, at the time of a great teachers' convention, gave much publicity to one of the so-called scientific studies in a way that was probably very deceptive to the public.

Let us examine two studies to illustrate this inadequacy. Consider first the most recent study by Adams in which he tried to discover the mathematical uses in general reading. He finds very little trace of algebra, geometry, or trigonometry.

In discussing the last item Bobbitt* concludes that therefore it is unwise to teach the simple materials of geometry and algebra in the junior high school years as is advocated by certain leaders. How much weight should be given to this advice is suggested in the next study.

The Frequency of Mathematical Terms, Especially Geometric Terms in Reading Materials.

All mathematical terms appearing on certain pages of some magazines† were tabulated. The following table indicates the items in order of frequency. The figure following each item shows the number of times the term was found. A word was tabulated only when used in a geometric sense.

*See Elementary School Journal, October 1924.
†The magazines were (1) Popular Mechanics every issue appearing in 1921, except the September number; (2) Popular Science every issue appearing in 1921 except the February, October, and November issues.

end	507	contact	24	point of contact	6
length	285	series	22	sector	6
side	245	extend	20	deflection	5
center	203	capacity	20	division	5
lines	201	triangle	20	extension	5
surfaces	185	radius	19	graduation	5
position	181	span	19	inverted	5
point	166	opposite side	19	inclination	5
cylinders	149	shaft	18	leg	5
space	130	located	18	perpendicular	5
diameter	128	straight	16	polygon	5
base	116	semi-circle	15	quadrant	5
edge	113	tapered	15	rectangular	5
sections	112	range	15	analysis	4
angle	109	location	14	axis	4
width	105	degrees	14	curvature	4
heights	103	solid	14	cubical	4
square	101	segments	14	perspective	4
measures	91	succession	14	projecting	4
distance	90	directed	13	rectangles	4
plane	80	laterally	13	reflecting	4
vertical	74	longitude	13	semicircular	4
thickness	73	altitude	12	variation	4
ratio	62	centrifugal	12	vertically	4
proportion	62	interior	12	adjacent	3
revolved	57	cylindrical	11	apex	3
depth	56	pitch	11	approximate	3
project	54	volume	11	coincide	3
horizontal	49	cycles	10	coincident	3
curved	46	grade	10	constant	3
level	44	ring	10	integral	3
faces	43	triangular	10	conclusions	3
circular	41	alinement	9	concave	3
area	40	alternating	9	concentric	3
opposite	38	angular	9	equi-distant	3
central	38	equivalent	9	longitudinally	3
similar	36	focus	9	longitudinal	3
parallel	35	latitude	9	mile	3
circle	35	pyramid	9	multiple	3
direction	35	linear	8	spirally	3
arcs	32	opposite end	8	terminates	3
divided	32	rotating	8	angle of inclination	2
equal	31	slope	8	coordinates	2
circumference	31	conical	7	cross section	2
cone	30	elevating	7	caliber	2
inclined	29	forms	7	diagonally	2
alternate	28	convex	6	differences	2
dimensions	28	deviation	6	directional	2
rotates	27	elevate	6	diagonal	2
corresponding	27	feet	6	definitions	2
scale	27	hexagonal	6	elliptic	2
arms	26	incline	6	ellipses	2
right angle	26	intervals	6	encircle	2
pivot	26	middle	6	equal parts	2

extremity	2	acute angle	1	logarithmic	1
graduate	2	breadth	1	lengthening	1
graph	2	cubic foot	1	magnitude	1
hemispheres	2	concentration	1	mid-way	1
hypothesis	2	cycloidal	1	mid-position	1
internally	2	corresponding sides	1	oblique	1
intersection	2	converged	1	octagon	1
limits	2	compass	1	opposite directions	1
long	2	chord	1	prisms	1
maximum	2	clockwise	1	proposition	1
minimum	2	cylinder	1	parallelism	1
meter	2	depends on	1	pentagon	1
mid-point	2	diagram	1	perimeter	1
polar	2	disproportionate	1	pi	1
projection	2	equilateral	1	plotted	1
reverse side	2	triangle	1	radiation	1
result	2	elongated	1	reflection	1
ring of surface	2	erected	1	reflective	1
rotation	2	equatorial	1	rotatable	1
spherical	2	elevation	1	square root	1
spindle	2	exterior	1	segment of a circle	1
successive	2	external	1	sequence	1
slanted	2	elliptical	1	straight side	1
segmental	2	foundation	1	terminals	1
square inch	2	geometry	1	transversally	1
taut	2	hemispherical	1	trajectory	1
transversal	2	hypotenuse	1	traversing	1
unit of measure	2	heptagon	1	tangent	1
vertical	2	hexagon	1	traversal	1
position	1	involute	1	triangulation	1
wide	1	identical	1	T-square	1
anti-clockwise	1	infinity	1	unequal	1
axial	1	intersect	1	whole	1
approximation	1	isosceles triangle	1	zero	1
altitudinous	1	lengthwise	1	zone	1

Conclusions:

1. It is important to note the larger number of geometric terms. In all there are 211 terms geometric in nature. This statement is of great importance in the light of Adams' investigation. In his study of the use of mathematics in general reading, Adams included only newspapers of the popular kind. Our study suggests that if you vary the material, keeping well within the range of what a considerable number of high school people read, a large number of geometric terms will be obtained.

2. The material in the preceding table probably has consid-

erable stability. When approximately three-fifths of the number of pages had been tabulated, the twenty highest items were

TABLE A

THE TWENTY HIGHEST ITEMS WHEN APPROXIMATELY 3/5 OF THE MATERIAL IS TABULATED

End	300	Center	86	Width	55	Diameter	63
Length	150	Surface	86	Distance	49	Height	63
Side	137	Point	85	Square	49	Section	60
Position	111	Edge	73	Angle	46	Cylinder	57
Lines	90	Base	68	Thickness	43	Space	56

ranked in order, as is shown in Table A. When all the material had been tabulated, the twenty highest items were again ranked, as is shown in Table B. It will be observed that only one item (thickness) is replaced by a new item (measures) when all the material is read.

TABLE B

THE TWENTY HIGHEST ITEMS WHEN ALL MATERIAL IS CONSIDERED

End	507	Surface	186	Space	128	Width	105
Length	285	Position	179	Base	116	Height	103
Side	245	Point	166	Edge	113	Square	99
Center	203	Cylinder	149	Section	112	Measure	91
Line	199	Diameter	128	Angle	109	Distance	90

3. It should not be inferred that the material constituted a random sampling of the material of this type that young people read. In fact, the terms "zero," "isosceles triangles," and "perimeter" all were found on the very last page read. No thoughtful person can observe the familiar terms appearing only once in our table without recognizing that the study needs to be extended before it is of much use as a guide. This fact does not reduce the importance of this study to zero. It is entirely adequate to show the limitation of Adams' selection of material. Moreover, it suggests the fallacy of Bobbitt's conclusion in the discussion of the study by Adams implying that high school pupils in their reading do not need to know geometric concepts.*

*Elementary School Journal, October 1924.

Next, let us examine what is probably the best known of the studies, namely Wilson's collection of problems. Wilson assumes that he has a random sampling. He found not a single trace of stocks or bonds. At the time that he made this study, according to government reports there were over 4,400,000 stockholders of corporations. It has not been possible to find an estimate of the number of bond holders, but it must have been very great, for municipal bonds, township pike bonds, and the like, had been in common use long before Wilson's study was made. There is here then, clear evidence that he did not have a complete picture of mathematical needs, but for the sake of argument let us make the false assumption that people did not in common life possess stocks and bonds at the time he collected his problems. Before Wilson analyzed his problems and printed his conclusions the Government was making a nation-wide drive to get all people to invest in Liberty Bonds. Mathematics must not only meet present needs, but it must teach those fundamental principles that will make it possible to anticipate future needs.

A third factor is one suggested by Thorndike. The question, "What uses do people make of mathematics?" is not the only issue. It is perhaps even more important to inquire: "What use could they make with great economy and profit?"

From these illustrations it is quite clear no single study can serve as a sole basis for choice of materials. Nevertheless the writer believes that the preceding summary of all the objective studies now available constitute a very valuable side light on our problem and which we dare not neglect if we wish to proceed intelligently in the choice of materials of instruction.

IV. THE PSYCHOLOGY OF DRILL IN MATHEMATICS

The National Committee on Mathematical Requirements with the assistance of nearly a hundred mathematical organizations undertook a national campaign to decrease the amount of manipulation of symbols and to increase the amount of

purposeful work with verbal problem materials. This program with few exceptions met with hearty approval in scores of mathematics meetings. In fact it is difficult to find anyone who does not approve of the Committee's effort *in theory*,—but practice is another matter. Many teachers traditionally trained and accustomed for many years to teaching courses involving probably eighty per cent of meaningless manipulation miss the complicated factoring problems, the "apartment house" fractions, and the "nests" of parentheses, as well as other old friends. They beg for more drill materials when they undertake to carry out the program suggested by the National Committee. We do not need more drill materials of the traditional type but more drill of the right kind. The data presented in an earlier section constitute clear evidence that what we have in the traditional courses does not function to any considerable extent, and it is not likely that we could get appreciable increase of skill by increasing the amount of drill. A physician having given an overdose of the wrong medicine would probably not decide to increase the amount.

While our traditional texts have placed much emphasis upon drill they have not to any considerable extent used the available guides for making drill materials effective. Perhaps the reason is that the accepted principles are nowhere listed in convenient form for the class room teacher. The busy teacher in reading his psychology is wearied by the necessity of going over sacks full of chaff which contain only a few handfuls of grain. The writer finds in the following list the principles which have been of practical help to him in the organization and administration of drill materials.

1. *Drill to be effective must be individual.*

We must stop drilling the few at the expense of the many and permit pupils to progress according to their varying skills and abilities. In ordinary practice we still teach a group of children as if all were at one level,—as a class. The

fact of course is that we are dealing with 25, 30 or 35 *individuals.* We would not permit a dentist, a physician or a surgeon to deal with children in mass fashion. We know the mental differences to be very great and to represent very different needs. Because of our necessity we hear much about the Dalton plan or some modification of it. We seem to have two groups, the first who cling fondly to the unmodified traditional group procedures. It is said that in such classes there are many failures or that many pupils are passed on to the next course with very low mastery. This indictment seems plausible for the teacher does not really know what is happening when he makes no analysis of his work and deals only with surface matters. In the other extreme there is a small group who advocate abandoning all recitations and making every part of instruction individual in character. Possibly there is truth in both contentions. One might argue that concepts, meanings, and attitudes can best be achieved with the momentum of a socialized or group recitation, but that the skills are best taught on an individual basis. Colvin estimated that the American secondary school wastes about one-half of its time. Is not this waste largely due to the senseless effort to fix the skills by group recitations?

2. *In general there should be much practice for a few skills rather than a little drill on each of many things.*

The objective evidence on the mastery of our pupils implies a need for decreasing the number of things to be expected of children. Can the teacher of physics honestly claim to teach all the elements that are designated important in our school texts? Many of the text books in the social studies are written at a dead level and to most teachers one item is as important as the next one. In arithmetic we spend much time in adding, subtracting, multiplying, and dividing the uncommon fractions to the neglect of those fractions which do occur frequently. It would be wholesome if high school teachers were to strive to teach fewer things but to teach them better.

3. *A drill exercise must be specific.*

It is common practice for teachers day after day to place miscellaneous exercises before the pupils. The teacher does not know which pupils need the drill nor does he know what particular bonds are strengthened by such exercises. If we wish pupils to learn how to place a decimal point, then it is possible for us to organize a series of problems in which the one element to be learned is placing the decimal point correctly. If we desire to give our pupils greater skill in learning to add halves, quarters, and eighths, then it is possible for us to set up a training series in which every pupil and the teacher will know that this is the specific thing to be practiced and similarly for every other bond that needs to be formed in mathematics. There is of course no implication here that we do not need miscellaneous lessons for recall and test purposes.

4. *A drill exercise must provide a scoring technique so that the pupil may watch his daily growth.*

One of life's greatest challenges, or interests, is *achievement.* Golf and tennis would for most of us be very stupid games if we had to play them without keeping scores. Teachers have always emphasized, perhaps too much, the idea of competition. We try to make each pupil excel others in the same group. For many pupils heredity and experience have made it impossible to do much in the way of exceeding others in the group. But every pupil in a class room can excel his previous record and this is probably as good a learning challenge as exists in a class room. It is to be hoped that this neglected means of motivation will be more largely utilized in the future.

5. *A drill exercise should be standardized.*

The pupil should have at least a rough notion of the task. It is helpful to the pupil to know what percentage of the pupils of his age or grade succeed in doing the tasks or at what level the work is to be done. Moreover the teacher needs this information for grade placement of exercises. It is inefficient for the seventh grade teacher to teach materials as if the

sixth grade teacher had never touched it. Supervisors insist that at times one can not tell by the material or method whether a class is a sixth grade, seventh grade, or an eighth grade. A city system should be able to state that it expects the excellent pupils in the sixth grade to be able to do a particular task at this *level of difficulty.* The implication is that the series of tasks shall be carefully chosen and shall have been given under precise conditions to many pupils,—in brief, it should be standardized.

6. *Drill material should be so constructed as to make possible the diagnosis of individual disabilities.*

Each pupil must have the opportunity to concentrate upon those processes which present peculiar difficulty to him. There is little drill material now available which enables teachers to do diagnostic work in any practical helpful way. The crux of the whole matter is the present machinery of test materials. Whether we continue to teach classes or go entirely to some individual basis we obviously need test materials that are more easily administered. The preparation of most teachers makes a simple organization imperative. It is believed that it is practical and feasible to construct a series of timed tests for a single grade with a single time unit (or multiple time units to be explained presently) for *all* drill units. Probably the optimum time for complete, concentrated work is close to three or four minutes in the seventh and eighth grades, and may be as much as eight or ten minutes for the more mature pupils in the algebra of the ninth year. Certainly the time for concentrated drill in the seventh and eighth grade arithmetic is not as long as 20 minutes, a figure suggested by a prominent psychologist. By this device once the testing begins *all* pupils take the test on the first unit of instruction. The pupils succeeding on the first trial proceed to the next test on the following day. By Thanksgiving time it is conceivable that Mary Brown may be working on the seventeenth test and John Doe on the second. All that the

teacher does is to start the whole group on a timed test and to stop them at the end of the period.

If a pupil fails to pass a particular test on a particular unit of work in a reasonable number of trials, it is obvious that he needs special attention. If he makes normal progress, as for example passing a test after a half-dozen trials at home and three or four official trials under the teacher's timing, he may be allowed to proceed without wasting any of the teacher's time or energy. This machinery is further improved by test booklets in place of the commonly used cards. Under the card system teachers start enthusiastically in the fall to manipulate the practice cards and to keep the complex records but soon lose out in the mass of routine and thereafter confusion reigns. A test booklet with all record forms and norms under one cover to be kept on the teacher's desk, one for each pupil, or, better still, in the hands of the pupil, seems to promise greater efficiency.

7. ***In the early stages in the fixing of a bond, progress should be relatively deliberate.***

In the early steps we should teach for power rather than skill. Another way of saying this is that the pupil should be provided with rich experience or numerous and vivid illustrations before much drill is administered. This is probably one of the principles in the psychology of drill most neglected in the teaching of mathematics. Algebra has often been severely criticized as placing extreme emphasis upon the manipulations of symbolism which have little meaning to the pupil. Teachers of freshmen courses in the university find that pupils can factor and simplify complex fractions with considerable skill in spite of little understanding. Probably the cause of this unfortunate condition is that high school teachers drill on processes before children understand. We attempt to cut across lots in learning when the road to clear concepts may be rough and circuitous. The teacher having been over the road often assumes that what is clear in his mind is easy to grasp by the pupil. He leaves "gaps" in the learning process

when he should construct intervening steps by richer pupil experience. The development of the newer courses for the seventh and eighth grades in which a wider use is made of sense experience as a basis for the ideas of the algebra of the ninth year represents an effort to apply this important principle.

8. *Drill should be organized so as to prevent the use of "crutches."*

The writer has observed one of the most brilliant teachers in the research field of mathematics take time day after day in his university classes to solve quadratic equations by "completing the square." When asked why he stopped his development again and again to take this roundabout method, he replied that he had never received drill in the use of the formula method during his high school days. He knew the formula method well enough but to use his words, "It wasn't handy" for him. A few summers ago the writer in the midst of a demonstration lesson at Teachers' College discovered a boy who could not pass the multiplication test with elements similar to the following: 465

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In taking this test the pupil said,
"7 x 5,—5 x 7 is 35, 7 x 6—6 x 7 is 42, 7 x 4—4 x 7 is 28, etc." In the intermediate grades he had learned to "twirl the stick" so as to grasp it by the easier end. The result was that it took him longer to do a multiplication problem than most boys of his mental age and grade. This same boy had a second crutch,—quite common, namely, to write at the side, the figure which is to be carried in the next step. In long problems he had many figures written at the side and frequently chose from this group the wrong figure to be carried. The thoughtful teacher of mathematics will be constantly on his guard not to establish mental connections that will interfere with later progress.

9. *Not all bonds should be given practice until high skill is obtained.*

Perhaps this principal should be stated as a corollary of the preceding. A good illustration is to be found in the method of teaching the solution of the quadratic equation. The instructor frequently uses four methods—the graphic, factoring, completing the square, and the formula. If a teacher has twenty hours for practice, how should he distribute his time? Would it be wise to give five hours practice on each method? Would it be wise to drill until the pupils have skill in the use of each method? Test results suggest that what happens under these conditions is that pupils achieve mastery of none. If pupils are to use the formula method, it will be wise to give just enough experience with the other methods for understanding and to place the emphasis for high skill on the formula method. Teaching a process by two methods does not imply that pupils will have higher skill in dealing with that process.

10. *The goal of a drill exercise must be a reasonable one.*

This principle seems to demand a complete reorganization of our drill materials for the mathematics of the secondary school. It will be admitted that everything else being equal an exercise becomes more interesting when a specific goal is set before the pupil. "Can you do forty problems in eight minutes?" is a greater challenge than "Can you do forty problems?" A goal involving speed and accuracy based on norms carefully secured is sound motivation but the difficulty lies in that our pupils are not so classified as to make it a fair proposition to expect the same performance of all children under standardized conditions. With the hope of solving this practical problem, the writer in cooperation with others has been experimenting to standardize instructional material on "*x*, *y*, and *z* levels." A single test follows:

"Goal 28. A Completion Test

"Supply the missing terms so that the expressions will be perfect square trinomials, and give the binomials of which

these trinomials are the squares. In *every* case write the sign which accompanies the missing term. Be sure to note the sign in parenthesis when it is given."

1 a^2.....$+ b^2 = (\ +\ )^2$
2 $9\,a^2$.....$+ 4 = (\ +\ )^2$
3 $25\,a^2$.....$+ 16 = (\ -\ )^2$
4 $16\,x^2$.....$+ y^2 = (\ -\ )^2$
5 $4\,x^2$.....$+ 81\,y^2 = (\ -\ )^2$
6 81.....$+ 36\,x^2 = (\ +\ )^2$
7 $a^2 - 6\,a$..... $= (\quad)^2$
8 $16\,x^2 - 8\,x$..... $= (\quad)^2$
9 $25\,x^2 + 40\,xy$..... $= (\quad)^2$
10 $36\,a^2 - 60\,a$..... $= (\quad)^2$
11 $81 + 36\,b^2$..... $= (\quad)^2$
12 $4 - 20\,ab$..... $= (\quad)^2$
13 .....$- 10\,xy + y^2 = (\quad)^2$
14 .....$- 16\,a^2 + 64 = (\quad)^2$
15 .....$- 50\,xy + 25 = (\quad)^2$
16 .....$- 60\,xy + 36 = (\quad)^2$
17 .....$- 18\,a^2 + 9 = (\quad)^2$
18 .....$+ 28\,b^2 + 49 = (\quad)^2$
19 .....$+ 32\,m + 64 = (\quad)^2$
20 $\frac{a^4}{4}$.....$+ b^4 = (\ -\ )^2$
21 $\frac{x^2}{4} - 10\,xy$..... $= (\quad)^2$
22 $16\,a^2$....$+ \frac{25\,b^2}{4} = (\ -\ )^2$
23 $\frac{x^2}{16}$.....$+ \frac{y^4}{9} = (\ +\ )^2$
24 $0.01\,x^4$.....$+ 0.25 = (\ -\ )^2$
25 $\frac{y^2}{49}$.....$+ 0.49 = (\ +\ )^2$
26 $0.64\,x^2y^2$.....$+ 25 = (\ -\ )^2$
27 $\frac{a^4b^2}{4} + 3\,a^2b$..... $= (\quad)^2$
28 .....$+ \frac{2\,a^2b}{5} + \frac{a^4}{25} = (\quad)^2$

GOAL

x (Excellent)	27
y (Good)	24
z (Passing)	18

RECORD OF MY IMPROVEMENT ON TEST 28

TRIAL	1	2	3	4	5	GOAL REACHED
NUMBER RIGHT						

Note that a pupil can pass this test at the z level by doing eighteen problems, or he may persist after passing the test and achieve a y mark or even the x goal. It is obvious that the *differentiated* goals can not be made without getting the records of what pupils of varying abilities and at different levels do with the particular exercises concerned. Hence, the establishment of these norms means an enormous amount of

work, time, and energy, but a necessity which teachers of mathematics and possibly those of other subjects now seem to face.

11. *RIGHT practice makes perfect.*

It is a common saying that "practice makes perfect," a statement far from true. Many of us get considerable practice in handwriting but we are not exceptionally excellent penmen. A beginning golfer grasped his club with hands crossed. No amount of practice yielded great skill under those conditions. The competent athletic coach knows that he must be constantly on his guard against practice which sets up interfering bonds. As Colvin has so aptly said, "Wrong practice is worse than no practice."

12. *Errors should be corrected before habits become fixed.*

This principle may possibly be a corollary of the preceding law but it is one that needs to be emphasized. Probably in mathematics we violate its implication most seriously with reference to written work. Pupils are given exercises to do which are not keyed and hence not self-scoring. The busy teacher finds it a physical impossibility to correct and return the papers before the situation has grown "cold." Hence the pupil is given an enormous amount of practice on incorrect forms and the worst of it is that the pupil does not know which response is right or which one is wrong. The time to catch an error is at the moment that it first occurs.

13. *Everything else being equal, a skill that is fixed in its natural setting will need less repetition.*

The three important factors that determine the strength of a connection are (1) the emotional setting in which it is formed, (2) repetition, and (3) recency. The thoughtful teacher knows that he dare not found his case on repetition and recency alone. The problems of mathematics must be motivated and the most effective motivation is to take a problem in its real setting or create class room conditions to simulate the actual situation. It is then that learning is most efficient. It may need to be recalled at this point that

in our Inventory Test the items which showed high mastery are precisely those which received practice in the out-of-school life of the pupils. There are those who say that the mathematics which represents life's uses need be given very little practice because pupils will later readily learn these tasks. Indeed, Meriam goes so far as to say "The way to teach arithmetic is not to teach it."* By that we assume that he means (1) we teach many useless things which the student forgets and hence this part is a waste of time and (2) the arithmetic which the student will need in life is very simple and learned more effectively at the time he meets the problems. There is evidence to support this position. The ease with which adult immigrants who have not come through our instruction in the schools adjust themselves to the social-economic phases of arithmetic suggests that we may be wasting much time. At any rate, practice a skill in its natural setting and fewer repetitions will be needed to fix it.

14. *In establishing a skill try to avoid the stage of diminishing returns.*

The school has often given practice on material not required by the demands of life. We have taught children to spell words that they never use. We are now in a most stupid fashion, and on a national scale, using drill materials in arithmetic in the seventh and eighth grade which are limited to practice with whole numbers. While this material may be highly profitable in the earlier grades, it certainly is senseless to confine seventh and eighth grade children to drill with whole numbers alone when they obviously need drill with the frequently occurring common fractions, decimals, per cents, etc. We are, moreover, teaching the various processes of arithmetic with practice materials which put emphasis on speed and accuracy, but no one really knows the speed and the accuracy demanded in uses out of school. To attempt to drill pupils beyond the desirable limits of a skill is surely a waste. We have clear illustrations in pupils trying to learn

*See "Reorganization of Mathematics in Secondary Schools," pp. 257-261.

a foreign language when they have very little linguistic ability, or in pupils with little gift who waste precious hours in practicing music. We also have a clear example of the stage of diminishing returns in the case of the pupil who has failed a course in plane geometry several times. Pupils who have reached their limit on a school task should certainly be permitted to substitute something else in its place.

15. *School life should be staged so that all desirable activities will have pleasurable outcomes and all undesirable activities will eventuate in unpleasant results.*

Thorndike has emphasized the efficacy of satisfaction in establishing a habit in his statement of the Law of Effect. Certainly this is one of the great laws of learning. We find it exemplified in the training of animals. Habits may be fixed in an animal in one of two ways; (1) by the giving of a reward, or (2) by creating an unpleasant condition. The direct form which the application of this principle may take in the class room is the association of *success* with learning. There is no greater challenge or stimulus to learning. The pupil must know at every stage of his learning how well he is succeeding. In spite of this elementary truth we have carefully kept answer books away from our pupils in mathematics. Answers furnished on verbal problems where pupils are tempted to work backward from the answers may be very injurious but psychology points clearly to the necessity of supplying answers to pupils when practicing materials for skill.

16. *In fixing a habit a pupil must be given an attitude in which he becomes a student of his own growth.*

An investigator much interested in spelling recently visited a certain school. He talked on spelling to the pupils at assembly, in their home rooms, and in the recitation rooms. Standardized tests were given to pupils, parents, and teachers. Everybody was talking spelling for "spelling was in the air." The pupils became especially interested in their own improvement. They made charts of their growth. The outcome was an astonishing amount of improvement in spelling

in a very short time. The explanation probably lies in the interest which the individual pupil took in his own "growth curve." A man may have a temporary slump when he begins to analyze his golf game but he is thereafter a hopeful player.

The principle also has a bearing on cheating. In a preceding section, it was argued that answers should be furnished the pupils in drill materials. The inexperienced teacher may say that this leads to dishonesty,—pupils will cheat. The fact is that pupils do not cheat themselves. A fine old man, the soul of honor, in the early days of his golf practice when his ball had rolled into a particularly unfavorable position was observed to kick it slyly into a more favorable position. It was later noted that he had neglected to count the "foot stroke" in his total score. Could it be said that this grand old man was a liar and a cheat? Surely not, for as soon as he became interested in the growth of his own game,—in his daily improvement, he could be trusted to keep an accurate score. In like manner pupils at their tasks may be relied upon to use answers honestly as soon as they have become students of their own growth.

17. *Habits must be formed in the psychological order.*

There undoubtedly is a best order of the elements that are emphasized in any drill exercise. If we are about to teach the multiplication tables to a class, should we teach them in this order: 2's, 3's, 4's, 5's, 6's, 7's, etc., or in this order: 2's, 5's, 10's, 4's, 3's, 6's, etc., or precisely what order should we use? Thorndike has emphasized that the psychological order of teaching the multiplication tables is not to teach them at all *as tables*. Accordingly, we need to drill on the multiplication facts without particular reference to formal arrangement.

In all probability the reader learned his factoring types in the following order: the difference of two squares, the square of the sum, the square of the difference, next the type x^2+ax+t and finally the type ax^2+bx+c. Conventional algebras usually take up one type after another in this order and give much practice to fix that particular kind of factoring ability.

It turns out that the more desirable order is just the reverse order. If a teacher gives exercises in the multiplication of two binomials so as to give a clear notion of the algebraic expression ax^2+bx+c and then proceeds at once to factor this type *first,* all the other cases mentioned are merely "easy specials" of this *general* case. The experimental evidence points definitely to the fact that the reverse order is the better psychologically.

It will take much time and effort to discover which orders are the better, but the improvement in the teaching of mathematics in the immediate future will very likely lie at this point.

18. *Practice should be distributed in diminishing amounts and at increasing intervals.*

The importance of repetition has already been suggested. The trouble has been that text books teach a topic once and for all. There is seldom adequate or wise provision for recall. The common method is to teach a process by an enormous amount of formal drill material presented in one "chunk." Occasionally a text will have a small review section, usually in the appendix where it is seldom found. The abandoned "cycle method" of teaching arithmetic had much of sound learning in it. We need to come back again and again to a principle if we are to gain mastery. Experimental evidence also points definitely to the need of increasing the length of the intervals between successive recalls.

19. *The more interesting aspects that enter into a skill should be taught early.*

We have grievously sinned in this matter in our teaching of mathematics. Competent teachers are almost unanimously testifying to the greater interest which pupils have in the newer courses for grades seven and eight because they include such matters as the simple principles of geometry, graphs, statistics, trigonometry, etc., and best of all, these topics open up a wealth of new applications. We no longer need to say to a seventh grade boy when he comes with a problem which

challenges him for the first time, "You will have to wait until the eleventh grade (four years hence) and then we will study your problem." The new material that is going into the seventh and eighth grades is replacing topics which are more abstract and less interesting.

20.* *The less difficult elements should be given before the more difficult.*

Our algebra of the ninth grade has been unnecessarily difficult. It deals almost wholly with the manipulation of symbolism and includes a minimum amount of material dealing with personal relations, with manipulation of things, with motion, with gathering of data, and generalization. We have been trying by brute force to put the thinking of our young people at the very top of the ladder of difficulty without giving them a chance to climb the intervening steps through concrete experiences. We have been trying to teach them to jump to the top of the ladder instead of letting them climb. There is, moreover, great obscurity in determining what constitutes difficulty. For example, the early test makers discovered that pupils did not know the zero combinations and they concluded promptly that addition problems like $4+0$ or multiplication exercises like 3×0, are difficult. There are prominent writers who have not yet discovered the fallacy. It is true that children miss these but it is because little practice is given and not on account of the difficulty. A class can be taught the correct response to $4 + 0$ much more readily than the correct one to 7×9. When we consider the difficulty of the elements in algebra we are even more at sea. Nevertheless significant changes can now be made on the basis of this principle. There can be no question but that the three or four weeks' unit of trigonometry now being taught in the eighth or ninth grade is less difficult than the factoring cases which it helps to displace. There can be no doubt that when pupils are per-

*The application of these principles will greatly modify our practice material. For an illustration of the newer type see "Instructional Tests in Algebra, Adjusted for Pupils of Varying Abilities." Schorling—Clark—Lindell. George Wahr, Publisher, Ann Arbor, Mich.

mitted to discover experimentally that the sum of the angles of a triangle is 180 degrees and then obliged to write this fact algebraically by the formula $x+y+z=180$ that they are dealing with material that is relatively simple because it involves handling things, measuring, collecting data in a table, drawing conclusions, and translating the conclusions algebraically. These thinking processes involve types of learning far simpler and more concrete than the solution of a quadratic equation or the simplifying of a complex fraction. Hence practice in these matters should be given at an early stage.

This concludes the discussion of drill. While the list may not be complete, it is hoped that enough has been given to illustrate to the beginning teacher and to the teacher of the newer courses the principles of drill which need emphasis in order that the work may not become vague and indefinite.

V. GENERAL MATHEMATICS

Two opposing points of view.

There are undoubtedly numerous educational theories. Almost every secondary school man has his own special philosophy. Nevertheless there are at this time two fairly well defined opposing points of view. The differences are suggested in the following columns:

	The Older Point of View	*The Newer Point of View*
1.	Essentially conservative.	In the main, progressive.
2.	Teach for skill.	Teach for power.
3.	Including much drill.	Wider use of sense experience.
4.	Approach to a new concept through the definition.	Approach to a new concept through varied illustration.
5.	Tending to the deductive method.	Tending to the inductive method.
6.	Formal language.	Language "geared" to the level of children.
7.	Believing that mathematics, Greek, and Latin have unique mental disciplinary values.	Believing that discipline is a function of the teacher; for example, a course in industrial arts well taught by a great personality may have more disciplinary value than a course of geometry that is poorly taught.

8.	Striving for completeness in the development of a single topic or subject.	Attempting to show the relationship between the different groups of materials.
9.	Emphasis on the logical.	Emphasis on the psychological.
10.	"Take it or leave it" attitude.	Insistence on motivation.

General mathematics has without question been the chief contribution of the newer point of view.

As is often the case in educational debates it may well be that the truth lies somewhere between these two extreme positions. It is the writer's opinion that it is possible to set up a program which will enable us as teachers to use the valuable outcomes of each point of view. This program has been previously presented in the five steps under the heading "remedial measures." There remains for us only to discuss the fifth remedial step, namely the teaching of general mathematics in grades seven and eight and in the ninth grade to these special groups: (1) those who for one reason or another are likely to have difficulty with mathematics; (2) those who are likely to drop out before graduating from high school.

Twenty-five years ago there were, so far as is known, no students studying courses in general mathematics. Today the pupils enrolled in courses in general mathematics are numbered by the hundred thousands. It must therefore be admitted, even by those who are still skeptical, that the rapid growth of general mathematics constitutes one of the striking changes of our times.

What is general mathematics?

General mathematics is an introductory, basic, exploratory course in which the simple and significant principles of arithmetic, algebra, geometry, intuitive geometry, statistics, and numerical trigonometry are taught so as to emphasize their natural and numerous inter-relations. There has been some confusion of terminology. Few people, for example, would venture to draw a distinction between general mathematics and correlated mathematics. However, there are exponents of the general mathematics movement who insist that there are at least two important differences. First, that general mathe-

matics is not a desirable substitute for demonstrative geometry, and second, that the value of general mathematics depends upon what is correlated and how it is correlated, rather than upon the fact that something has somehow been correlated. It is argued, moreover, that the movement started by Professors Moore and Myers put far too much emphasis upon correlation for the sake of correlation. General mathematics has used the correlation idea,—the emphasis upon inter-relations. But its chief *objective* has been *wholly different* from that of correlated mathematics. The aim of general mathematics is close to that expressed in the document "Cardinal Principles of Secondary Education." The movement, started about ten years ago, represented an effort to get a course in the ninth year which would more nearly meet the needs of pupils, particularly those of low ability and poor background and those who would leave school before graduating. There was no intention to set up three or four years of college preparatory courses. The emphasis was on the notion of a basic exploratory course. For evidence the reader may turn to the preface of the earliest books written on general mathematics. At that time it was not possible to teach such a course in the seventh and eighth grades to any considerable extent because secondary school people didn't control the seventh and eighth grades. With the rapid growth in the junior high school movement the opportunity to give such a basic course now lies in the seventh and eighth grades.

The aims of a general course in mathematics.

If one glances through the literature of general mathematics it is noted that the following outcomes were expected as the results of organizing such courses: (1) on the side of subject matter, (a) control of the simple and important parts of algebra and geometry, (b) an introduction to trigonometry and statistics probably adequate for common needs, (c) greater facility in the use of fractions (common and decimal) and of percentage relations, (d) training in the use of a number of "optional" topics, e. g., logarithms, the slide-rule and tables;

(2) the considerable reduction of the number of first year failures; (3) increase in the number of pupils taking a subsequent course in plane geometry with interest and profit; (4) more intelligent election of later courses in mathematics; (5) more adequate preparation for the mathematical needs of other school subjects, industrial arts, household arts, physics, chemistry, and the like; (6) a beginning on the part of the student in the technique of investigation (many parts of the material are organized in laboratory form); (8) greater power in problem solving through the use of more methods of attack (correlation is used as a means and not as an end); (9) a clearer notion of the relationship of various mathematical methods without a forced correlation (there is no attempt to correlate plane geometry with material from other fields); (10) a better understanding of algebra as far as it goes (this is probably well within the limits of ordinary life needs); (11) an appreciation and understanding of the importance of the idea of relationship (function concept) and; (12) greater enjoyment in the study of mathematics.

Conditions favoring the growth of general mathematics.

Among the factors requiring the extension of general mathematics courses are the following:

(1) The population of our high schools has in the last quarter of a century been multiplied by more than ten whereas the population of our nation has not doubled. This increase in school population has given us a wider sampling of the general public and hence has in all probability lowered the level of ability. Certainly it has given us pupils with less background to do the conventional courses successfully. The language difficulties which the teacher confronts in instructing the children of recent immigrants,—a problem met in many high schools,—is alone very great.

(2) The number of children who should take one-year courses in mathematics is very large. There are cities of considerable size in Michigan in which eighty per cent of

the children still drop out of school before entering senior high school. The mathematical materials that should be emphasized are computation and the simple principles in algebra and geometry which they will need in general reading, in the shops, and in the commercial pursuits.

(3) The investigation conducted by the National Committee on Mathematical Requirements and reported on page 45 in the "Reorganization of Mathematics in Secondary Education" indicates clearly that what the college man or woman needs to know are precisely those elementary principles which a half dozen series of junior high school text books are striving to teach with great emphasis. In the report of the Committee we read, "It is interesting to note how closely the modifications suggested by this inquiry [on college needs] correspond to the modifications in secondary school mathematics foreshadowed by the study of needs of the high school pupil irrespective of his possible future college attendance," and later we read, "That they should be in such close accord with the desires of college teachers in the fields of physical and social sciences as to entrance requirements is striking." A caution may not be out of place at this point. Test results, as well as common sense, tell us that pupils forget mathematics and all other things very rapidly. If a pupil takes a course in general mathematics (or algebra in the ninth grade) and no mathematics in all the years that intervene between this and the time of college entrance, he must not be expected to know much mathematics. Several practices now exist to solve this problem. (a) The competent pupils are sorted out at the beginning of the ninth school year and given algebra in the ninth year to be followed later by geometry and more algebra. (b) The pupils who, after taking a year of general mathematics, desire to enter college are enrolled in second semester algebra and from that point on travel the usual route. (c) Pupils may take their mathematics courses in the later years of their high school work. This puts their preparation in mathematics nearer to their college entrance. The last is but little found in practice because many pupils needing the course would be

leaving school before they had a chance to take the course.

(4) The rapid growth of the junior high school movement has given teachers a greater opportunity to teach worth while courses in the seventh and eighth grades. To be sure there is no reason why the same course could not be given by a competent teacher in the seventh and eighth grades of the conventional elementary school. But the fact is that educators are more anxious to initiate solid, substantial work in these grades once they have accepted the machinery of the junior high school.

(5) The large number of failures together with the very low mastery on the part of pupils who pass the courses, makes it necessary that we organize the materials in the form in which they are most readily learned. General mathematics utilizes a wider range of sensory experience. Everything else being equal, a problem accompanied by a graphic picture is more easily understood and appreciated by a greater number of pupils. Can "easy" mathematics be worth while? The psychologist says that a subject can not be made too easy. Surely the teacher who by keener insight into the nature of mental life succeeds in making mathematics clear and vivid is to be preferred to the teacher who by the lack of psychological insight presents the same subject matter in a way that makes it difficult for pupils. There is evidence that a large number of pupils in our high schools do not have the ability necessary to pass the formal algebra course,—at any rate not without some preparatory course along the lines of general mathematics.

(6) General mathematics facilitates motivation. Because the logical order has been replaced by the psychological order, in which the pupil learns the important principles of algebra and geometry along with his arithmetic much earlier, the opportunity for richer application is present throughout the junior high school years. If a pupil becomes interested in a problem in another school subject or in his out-of-school experience, he need not be required to wait three or four years before he can understand its solution. Since

the materials are drawn from several different fields, the illustrations are more varied. Hence the general courses are said to be more interesting for most pupils.

Conclusion.

This discussion has recognized the *improvement of the quality of scholarship* as the most important problem arising out of the last quarter of a century. Two points of view concerning the teaching of mathematics have been presented. These two positions appear to be widely separated. It has been argued that each side has something of value in it for us that we dare not neglect. In fact a program has here been proposed which would utilize the chief values from each group. The suggestions for improving the quality of scholarship that have been discussed in some detail are:

1. The formulation of a basic philosophy (from the newer education).
2. The listing of specific objectives (from the older education).
3. The placing of emphasis on objective studies in the choice of objectives (from the newer education).
4. The careful use of the psychology of drill (from the older education).
5. The teaching of general mathematics in grades seven and eight and for a limited group in the ninth school year when pupils have not had these materials in grades seven and eight (from the newer education).

Why do we always need to choose between the "old" and the "new"? The old is never dead and the new never altogether new. The leaves of a tree fall to the ground, wither and fade only to live more vividly in the new. Not the old *or* the new, but the old *and* the new.

IMPROVEMENT OF TESTS IN MATHEMATICS

By W. D. Reeve

INTRODUCTION

It will be admitted that the testing movement is wholly the product of the last twenty-five years. Hence, there is a place in this Yearbook for a section of which the aim is to present a brief review of the various types of tests in mathematics that have been used in this country during the past generation, to discuss the advantages and disadvantages of each, to point out the progress that has been made in such tests, and especially to recommend a better use of tests for the future. The discussion is concerned for the most part with tests of algebraic abilities.

It is only within the last generation that any attempt has been made to bring about genuine reform in the method of measuring the effectiveness of instruction in secondary schools. The main burden of the section will be to recommend a new type of testing program where the emphasis is placed upon a method of procedure in testing rather than upon any single test itself.

Prognostic Tests

Tests in mathematics may be classified under two main heads: namely, prognostic tests and achievement tests. Prognostic tests are those which are given early in the pupil's career for the purpose of measuring his innate ability to do mathematical work. By their use a teacher is enabled to predict the probable success of a pupil in his later work. For

example, Professor Rogers' prognostic* tests were designed to measure the specific ability of pupils to succeed in studying secondary mathematics.

Prognostic tests were developed to meet the need for some kind of instrument to measure a pupil's capacity to learn mathematics, in order that he might be more intelligently advised with respect to his subsequent work. Since previous tests of this type, except those relating to geometry, dealt only with the more mechanical phases of the work, mathematicians objected to them on the ground that there is little in common between the habits of symbol manipulation, which was emphasized by nearly all the early tests, and the more fundamental processes of reasoning which are characteristic of higher mathematical study. In general they maintained that the intensive tests in special fields like arithmetic, algebra, and geometry failed of their purpose for the reason that these subjects are rarely applied by themselves. It should be observed, however, that everything depends upon what purpose the maker of a test intends it to serve. If it is not intended to be prognostic, there may be justification for such a test on some other basis.†

No one seems to have claimed that prognostic tests can be, or should be, the sole basis for prediction. Nevertheless, they constitute an important step in the direction of trying to discover those pupils who give the most promise in the field of mathematics. If such tests can be of assistance in helping us in our problem of discovering individual differences in mathematical ability, they may become useful instruments for prognosis in our schools. It should be insisted, however, that they must not be made the sole basis for prediction, classification, or guidance, because such absolute dependence would overlook other important factors. Besides, there is no prognostic test available today whose use will enable us to predict success in mathematics any more than the prophesy

*A. L. Rogers, Tests of Mathematical Ability and their prognostic Value, *Teachers College Contributions to Education*, No. 89.

†B. D. Wood, Measurements in Higher Education, World Book Co., 1923.

of success we can obtain from any good test of abstract intelligence.*

In a recent article† Professor David Eugene Smith says: "The prognostic test at its best achieves quickly and with improved results that which the schools have heretofore discovered after a loss of valuable time; at its worst it leads into a determinism that is more dangerous than the extreme form of Calvinism which left each individual absolutely without hope. On the whole the tests have achieved a great and well-deserved success, and this success will be much more apparent when a new generation comes forward to correct the errors of the present one."

If such tests can be made in such a way that they may become a reliable help in prognosis, we should all favor their wider use in selecting those pupils who are most likely to profit by advanced work in mathematics. In that case we might and probably should, require mathematics only through the ninth year of the junior high school, offering the subsequent courses only to those who are capable of a much higher grade of work than is now possible. Such prognostic tests would then be the means of doing justice to a larger number of pupils who are forced to study mathematics against their will and in many cases without much gain, or at least they could be used to advantage in determining what kind of mathematics would be most valuable to a given pupil. If we had in the tenth-year, eleventh-year, and twelfth-year classes in mathematics only those pupils who liked the subject and were able to learn it, the joy that would come to teachers and pupils alike would more than offset the loss in numbers that would result from the adoption of such a program.

Achievement Tests

While prognostic tests are concerned with ascertaining what

*See E. L. Thorndike and Others, The Psychology of Algebra. The Macmillan Company, 1924, pp. 216-217.

†D. E. Smith, "On Improving Algebra Tests," *Teachers College Record*, March 1923, pp. 87-88.

See also D. E. Smith, The Progress of Algebra. Ginn and Company, 1925. pp. 39-48.

a pupil is able to learn, it is the function of achievement tests to determine what he has learned. Such tests are intended to measure the progress which a pupil or a class is making in a given topic or course. They also measure indirectly the pupil's innate ability, since whatever progress he makes is necessarily dependent upon this ability. These tests, however, are so affected by classroom influences that they cannot be accepted as reliable instruments for the accurate measurement of capacity to learn, although in general they correlate rather highly with intelligence. Both prognostic and achievement tests may rightly be considered diagnostic in nature, although we have thus far made little use of diagnosis in the former type. Furthermore, any achievement test may be made diagnostic even though it may not be so intended, and an achievement test may have diagnostic value and still not be used for purposes of diagnosis.

Types of Achievement Tests

Two types of achievement tests have had rather wide use in this country. The first type includes the tests set by the teacher alone or in cooperation with others who are more or less responsible for the existing course of study; the second includes the "extramural" tests set by examiners who have little or no direct contact with the classroom. Illustrations of the latter type are the ordinary College Entrance Board Examinations, certain well-known state examinations like the New York State Regents Examinations, and the so-called standardized tests or scales.

It is not the intention to minimize the importance of any one of the tests mentioned above, especially if it can be made to serve some useful purpose, but rather to emphasize a wider use of certain types and a more intelligent use of others.

All of the tests just mentioned except the standardized tests or scales belong to the class that is ordinarily known as "essay type" examinations.

Some of the modern writers recognize three types of achievement tests as follows:

1. The traditional "essay type" examination.
2. Standardized tests or scales.
3. The "new-type" objective examination.

The subsequent discussion will cover all of these various types.

Tests Set by the Teacher

An entire volume could be written on the subject of tests set by the teacher, but space will not permit any more than a statement of a few of their advantages and disadvantages. It is well known that many teachers are not qualified to make the careful analysis of a course in mathematics which is necessary to determine the abilities that are fundamental and those that can or should be measured. As a result certain teachers include in important tests a meager sampling of the large range of abilities to be developed and regard these few questions as an instrument for measuring the entire field.*

It is not as easy to make a good examination as many teachers seem to think. It takes a great deal of time, energy, and thought to construct a suitable test, but there is probably no part of our work today where more important results would follow than that of taking greater interest and care in making satisfactory tests. We have known for a long time that our examinations have been inadequate, but little has been done about it.

Because the main purpose of diagnosis is to guide the teacher in remedial instruction it is purposed later in this discussion to emphasize the importance of diagnostic tests as instruments for aiding the teacher in improving instruction. Without some plan of discovering the particular defects of a pupil or a class the work of the teacher is likely to be more or less futile.

If the proper kinds of tests are given, the progress made by pupils can be measured, and the norms of accomplishment thus established can be used by the teacher as a guide in discovering the educational needs of future classes. In addition to this

*B. D. Wood, Measurement in Higher Education. World Book Company, 1923. p. 161.

the pupils assist the teacher greatly by locating not only their particular difficulties but the causes of these difficulties as well. The assistance rendered by the pupil's introspection in diagnosis should not be overlooked, because a pupil will often find the cause of a difficulty more quickly than the teacher.

In the second place, with many teachers the marking of test papers is highly subjective; that is, the mark given depends to a large extent upon the person who does the marking.* Furthermore, the measures obtained by the teacher from many of the ordinary "essay type" examinations are not truly diagnostic; that is, they do not point out the particular details upon which the pupil is either weak or strong. It is thus clear that if specific measures for diagnosis are desired, the ordinary examination will not prove entirely satisfactory. Instead of continuing the practice of doing injustice to thousands of pupils every year because our tests are misused, we must in some way or other develop a new marking system which will enable us to rate their achievement in relation to their ability.

In spite of the frequent inadequacy and inaccuracy of teachers' judgments, both in setting good examinations and in marking them fairly, it should be more generally realized that these same teachers are in the long run the ones best qualified to do the task. They learn not only how to make objective tests that will have both measuring and diagnostic value, but they can also learn to use them intelligently. This ability to use the test will increase in proportion to the progress they make in understanding scientific methods of measurement. Instead of constantly reminding them of their failures, without making suggestions for improvement, attention should be directed to the traditional method of marking rather than to the teacher who, because of the lack of a better method, is forced to use one that is neither exact nor reliable. School marks have long been used to measure a multitude of things which have not and cannot be measured with any degree of

*D. Starch and E. C. Elliott, "The Reliability of Grading Work in Mathematics," *School Review*, April, 1913, Vol. XVII, p. 254.

precision.* By a proper development and use of objective tests we may be able to work out a better marking system for the future.

Tests Set by Persons Other than the Teacher

During the past generation there have been many discussions regarding the examinations set by state and college authorities. Some progress has been made in the content of these examinations and in the method of conducting them, but much remains to be done before they can be said to have reached their highest development even for the purpose for which they are designed. Speaking of such tests, Professor David Eugene Smith† says: "New York State has dictated by regents examinations, generally good for a poor teacher and generally bad for a good one. The College Entrance Examination Board has also, and naturally, dictated what should be taught in algebra, and has recently made a long step in advance by a series of improvements. Each of these cases of dictation has contributed powerfully to making algebra stagnant, and each has been potent in keeping it on a dead level of traditional mechanism."

Professor Smith has implied in the preceding statement that the people who formulate the tests referred to above are often better able to judge what is fundamental in a course than are some classroom teachers of mathematics. In fact, the recent College Entrance Examination Board syllabi reflect more modern types of mathematics courses than those which a great many teachers are now using.‡ The trouble is not so much with the original purpose of such examinations as with their imperfect use and interpretation. In many cases these examinations measure abilities which have not been considered in the previous instruction of many of the teachers whose pupils have to take the examinations, and consequently the

*W. A. McCall, How to Measure in Education. The Macmillan Company, 1922. p. 59.

†David Eugene Smith, "On Improving Algebra Tests," *Teachers College Record*, March, 1923, Vol. XXIV, p. 87.

‡See the Report of the Secretary of the College Entrance Examination Board for 1921, pp. 1-4, also for 1922, pp. 18-19, and for 1923, pp. 1-3 and 7-9.

results secured cannot be called fair measures of the teachers' effectiveness. Nevertheless it is true that the efficiency of teachers and the efficiency of classes have been compared on the basis of the scores made on such examinations. As a result all sorts of methods are resorted to by certain teachers whose sole aim is to get their pupils through these examinations with a passing grade. Such practices, it need hardly be said, do not represent a desirable type of modern education. Since these examinations, as now regulated and administered, have not given satisfactory results,* the question arises as to whether or not it would be a wise plan to replace them with intelligence tests or prognostic tests of some kind that can be used as a basis for determining what pupils are able to carry on higher mathematical work.

The mere fact that so much discussion† has gone on recently with reference to the value of "extramural" examinations‡ indicates that many thoughtful people are not yet satisfied with the present status of such tests.

Standardized Tests

The so-called standardized tests have recently come into rather wide use in this country, but their introduction has been accompanied, in a great many places, by grave misuses. Sometimes those in charge of the testing program are so unwise as to judge the success or failure of a teacher solely by the outcome of certain tests. In some cases invidious comparisons made on the basis of test results have not been checked by careful study of the methods of teaching employed. In still other cases certain standardized algebra tests have been regarded as entirely reliable, and it has been assumed that they could be used to measure every feature of the teaching of that particular subject.

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*B. D. Wood, Measurement in Higher Education, Chapter VI. World Book Company, 1923.

†M. Barnes, "Procrustes Redivivus," *Atlantic Monthly*, July, 1925, p. 83.

‡W. M. Proctor, "The High School's Interest in Methods of Selecting Students for College Admission," *School and Society*, October 10, 1925, Vol. XXII, pp. 441-448.

Professor Thorndike says* "the testing movement is embarrassed by its success." He says further that "improvement in the instruments of measurement, both of intellect and of school achievement, is more desirable than multiplication of their number." It is proper to add that in a transition period of curriculum construction, like the one in which we find ourselves, we should be particularly careful to seek only an intelligent use of all tests.

A great deal of the traditional algebra which has been questioned by many teachers and textbook writers as contrary to modern objectives has been replaced by more valuable material. It is an interesting fact, however, that standardized tests have failed to include such material and thus have made difficult the general acceptance of some of the more modern algebra courses. The values claimed for standardized tests because of their carefully selected content have not always been legitimate. To quote Professor David Eugene Smith† again: "The complaint is not so much that the tests are solely mechanical, involving only a minimum of intellectual processes,—a fault that is probably inevitable in the present stage of development, but which is being successfully removed in some of the arithmetic tests; it is also that the material required for testing the mechanical processes is often such as should play only a minor role, if any, in the education of the average citizen. The tests represent generally a dead level of dull grind, offering to the teacher only this ideal of an algebra course. He may escape from the curriculum, making his own course; he may and should select from the textbook that which he needs for carrying out his plan; but he cannot escape from teaching those things that are required by outside examinations whether they be set by boards of regents, by colleges, or by educational testers." Professor Smith goes on to point out a

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*E. L. Thorndike, "Standard Tests and their Use—A Symposium," *Teachers College Record*, October, 1924, p. 93.

†D. E. Smith, "On Improving Algebra Tests," *Teachers College Record*, March, 1923, Vol. XXIV, p. 88.

See also D. E. Smith, Progress in Algebra, Ginn and Company, 1925, pp. 39-48.

large number of examples taken from recent standardized tests in algebra, some of which he commends, but many of which he condemns, and rightly so. For example, he quotes the following exercise from one test:

$$\text{Divide} \quad \frac{\sqrt{2}}{\sqrt{3}} \quad \text{by} \quad \frac{\sqrt{2}}{3}$$

He then adds: "Why should anyone ever wish to perform this division? It is not algebraic, and it has no particular significance either in algebra or in connection with formulas that are likely to arise. It is possible to test our schools on something that we have some other reason to perpetuate than the mere reason for meeting this kind of a test."

Wrong Use of Standardized Tests

Then, as another puts it, "Too often tests are given, the data are tabulated, conclusions drawn therefrom are utilized by supervisors, and methods are revised by teachers because of them; but the pupils who wrote the tests are not informed of any of the results except in those rooms where unsatisfactory conditions have brought about attempts to shift the blame to these pupils." Here lies the trouble in many of the standardized tests. Teachers and pupils alike are often given little consideration in their administration, and the result is that great loss ensues in the improvement of instruction.

Professor Woody* has pointed out that "from one point of view standardized measurement represents the refinement obtained through the crossing of current practice and scientific method." No doubt many of our test makers have appreciated this point of view, but we might have made better use of some of their work if it had been more generally understood earlier.

Teachers always have measured and always will measure their pupils in some way or other, but the science of measurement will not reach its maximum of importance until the teachers and the makers of tests establish a partnership. Each may then hope to develop scientific methods of approach, and

*C. Woody, "Standard Tests and their Use—A Symposium," *Teachers College Record*, October, 1924, Vol. XXVI, p. 95.

only then will the teachers themselves have an abiding interest in the outcome of their work. It is not essential that teachers should construct tests themselves, although many will soon learn the technique of the process, but they should at least be consulted in order to assure a result that shall be generally useful to all concerned. It may be, as Professor Trabue has stated,* "that college professors and graduate students of education may continue to build and improve the measuring instruments used in testing school pupils, but the diagnosis of specific educational disease and the long experimentation necessary to discover whether for a certain type of pupil more drill on forms will be helpful or fatal,—the 'case work' which constitutes the next step in the use of educational measurements,—must be done in the classroom by the regular teachers."

A careful study of many of the standardized tests shows that they have been prepared by people who select material with little regard to the proper objectives to be obtained and who seem to be ignorant of the plans for the reorganization of mathematics. Moreover, there exist many desirable objectives which were overlooked when these tests were made. Statistically the tests themselves may be perfect; but when a task is introduced into a test merely because it represents a certain degree of difficulty, it simply tends to prejudice all persons of common sense against the whole movement. It is such unfortunate errors as these that retard progress.

The greatest use of standardized tests has been made in arithmetic. This has been due to the fact that the material is such as to be easily adapted for standardization. Even in this field the tendency today is away from general national standardization to practice exercises and diagnosis of individual cases. This has been due to a realization of the importance of giving the pupils their standing on some definite scale of performance related to their own class rather than to try to place them with reference to a "norm" based on the performance of some outside group—a practice due

*M. R. Trabue, "Standard Tests and their Use—A Symposium," *Teachers College Record*, October, 1924, Vol. XXVI, p. 95.

entirely to the influence of the recent testing movement.

Chief Values Obtained from Standardized Tests

The chief values obtained from the use of standardized tests in mathematics may be summarized as follows:

1. They have brought out more clearly the problem arising from individual differences in ability among pupils.

2. They have shown us that a great deal of the traditional material is too difficult for most pupils and therefore should not be taught.

3. In some cases they have also shown that certain material was easier than had been expected and that it can be learned by a majority of pupils.

4. They have made it possible for the teacher to stop drilling certain pupils beyond the stage of diminishing returns.

5. They have made it possible for us to develop certain standards of achievement which are clearly defined and which can be assigned to varying levels of intelligence. As we shall see, however, this value has occasionally been misunderstood.

6. They have contributed to the development of more objective methods of testing.

7. They have, when they have been intelligently used, stimulated pupils to renewed effort in trying to reach certain standards of perfection.

All of the above outcomes have been worth while, but it is equally true that standardized tests, as I have previously pointed out, do not lend themselves readily to some of the more important needs which an ideal testing program presents. Professor Upton* has given a thorough discussion of the influence of standardized tests on the curriculum in arithmetic. Professor Smith† has made certain criticisms of algebra tests and has offered suggestions for improvement. A great deal

*C. B. Upton, "The Influence of Standardized Tests on the Curriculum in Arithmetic," *Teachers College Record*, April, 1925, Vol. XXVI, p. 627.

†D. E. Smith, "On Improving Algebra Tests," *Teachers College Record*, March, 1923, Vol. XXIV, pp. 87-88.

of my own time in the past nine years has been devoted to a study of diagnostic tests, and the results‡ are a part of the basis for the recommendations made in this chapter.

The Future of Standardized Tests

It is not difficult to formulate satisfactory standardized tests that are more objective than the older types, that are more uniform with respect to administration and the scoring of results, that furnish desirable norms of achievement for guidance in future instruction, and that are much more reliable as measuring instruments. It is not certain, however, that we should attempt to emphasize the standardization of tests as measuring devices before we know what abilities we wish to measure. It would seem that at the present time we should be more interested in determining clearly the purposes in view in the teaching of mathematics, the content best fitted to help us realize these purposes, and the kind of tests that will afford a check upon our results. This does not mean that no measuring should be done in the meantime, but rather that our methods of measuring should be improved before we seek to increase the use of standardized tests.*

In all fairness to standardized tests, however, it should be said that they have gone beyond what Professor Woody calls the "curiosity" stage and "the stage in which the predominant idea was the use of the tests for determining existing levels of achievement," and, in some respects at least, have approached the third stage, "in which the predominant idea is the utilization of tests as a means for the improvement of instruction." They have been helpful in this third stage, however, only as they furnish facts concerning certain "levels of efficiency" reached by pupils, and thus "contribute to the evaluation and diagnosis of the efficiency of instruction."

Where standardized tests are valid and reliable instruments, they may be profitably used for purposes of "general-survey

‡W. D. Reeve, A Diagnostic Study of the Teaching Problems in High School Mathematics. Ginn and Company, 1926.

*W. S. Monroe, Theory of Educational Measurements. Houghton Mifflin Company, 1923, pp. 42 and 49.

diagnosis," and even in some cases for class and individual diagnosis; but this work must be based more and more on the cooperation of all concerned, from the superintendent down to the pupils themselves.

It seems reasonable to expect that we shall in due time have more tests that are prepared by scholars, those who know not only the underlying principles of these devices but also the true objectives in teaching mathematics. These tests can be made in such forms that they may be used not only for both general and particular diagnosis but also as aids in improving instruction. When this is done, the establishment of norms of attainment which can be generally used will follow. We should realize, however, that a teacher is not always justified in feeling satisfied with the results when pupils reach a certain norm of achievement. To do this is to overlook the fact that the standard norms may be raised by lifting the general level of achievement through better methods of teaching. On the other hand, "they need to see that the law of diminishing returns applies to educational products as well as to economic products, and that continually trying to raise the level of achievement may result in educational bankruptcy."*

It is reasonable to expect that we might reach our educational objectives without the type of test mentioned above; but it is hardly conceivable that we shall be able to succeed as we wish unless we develop, for general mass instruction, a type of teaching that is based upon the specific needs of individual classes and pupils. This sort of instruction should be remedial and must be based upon specific diagnostic measures of pupils' achievements.

It is well known that there are differences of opinion with respect to what is important to teach in mathematics in a given course.† Until we have a more carefully considered collection of tests, it is futile to expect them to serve any very use-

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*C. Woody, "Standard Tests and their Uses — A Symposium," *Teachers College Record*, October, 1924, Vol. XXVI, p. 104.

†H. C. Barber, Teaching Junior High School Mathematics. Houghton Mifflin Company, 1924, p. 134.

ful purpose. When that time comes, many teachers will consider them no less important than their textbooks. They will then use tests both as teaching devices and as measuring instruments in all their work. This will be a great improvement over some of our present practices where teachers who feel the pressure of tests set from without fall into the habit of making such tests their textboks.

In chapter 13 of the report of the National Committee on Mathematical Requirements Professor Upton gave a very careful and complete discussion of standardized tests in mathematics for secondary schools. His report included not only a description and discussion of the standardized tests in use since 1914, but also gave illustrations of specific tests in arithmetic, algebra, geometry and also those concerned with the measuring of general mathematical ability. Teachers who are interested in knowing more about the nature and content of such tests will find this chapter of the National Committee Report a very valuable source of information and help.

While standardized tests have been of real service in the ways that have been pointed out and therefore cannot be condemned if they serve the purpose for which they were intended, nevertheless there are reasons at present for giving them less prominence in educational discussions and for turning our attention to a testing program that seems to offer us an opportunity to obtain more important results. Nothing that has been found out by means of standardized tests is too difficult for us to discover with the kind of testing program which it is the purpose of this section to emphasize.

The Place of Tests in Curriculum Construction

Obviously, the most important, at least the most discussed, problem before us today is the matter of curriculum construction. The task of reorganizing mathematics so as to provide a desirable body of material for the junior and senior high school involves at least four main steps.

The first step is to set up a list of desirable objectives to be attained in the teaching of mathematics. The second step

is to determine the nature and the extent of the subject matter which will best enable teachers to realize the chosen objectives. The third step is to develop the best methods of teaching the selected subject matter. This cannot be done without some kind of testing program whereby the teacher is able to discover to what extent his methods are paying dividends. No matter how desirable the content may seem to be and how well his methods are perfected, it may be that the material is too difficult for the pupils in a given year. Moreover, it may be also true that even though it is possible for children to learn certain things in mathematics at a given age, the time for learning it is too great to justify its inclusion in the course of study for that year. We must, therefore, have a fourth step which is a testing program that will enable us to see how well the pupils are learning the things we have been trying to teach. This last step necessarily involves a more careful analysis than heretofore of how pupils learn most efficiently and easily.

Guiding Principles Underlying Good Tests

The following guiding principles underlie the construction of a good test in mathematics:

1. Every test should attempt to increase the pupil's ability to master only the subject matter that has been presented to him. This means that the teacher must discern clearly the objectives* in the topic or course and must build his examination so as to measure the extent to which these objectives have been realized. Such a procedure measures the progress made by a pupil or a class. This is set down by some as the first aim of an examination.† A test which has no reference to what has already been taught cannot meet this requirement. If pupils, who have the native capacity to learn a certain thing, fail to do so, there is a chance for remedial work to bear fruit. On the other hand, there is little to be expected or gained by

*W. D. Reeve, "Objectives in the Teaching of Mathematics." *The Mathematics Teacher*, November, 1925, Vol. XVIII, pp. 385-405.

†A. L. Lowell, "The Art of Examination," *The Atlantic Monthly*, January, 1926, p. 58.

remedial work in the case of pupils who have learned as well as can be expected considering their innate capacity. If both the diagnosis and the subsequent remedial work suggested by the tests are to be of value, the teacher must have at hand specific measuremnets of the pupil's achievement. Such measurements can be fairly made only in a field in which the pupil has been working.

Moreover, in improving instruction, tests that contain material that has not been taught previously cannot have much diagnostic value. In such a case it is impossible to tell whether poor results mean that the content of the course was too difficult or that the teaching was not well done.

2. Every test should emphasize those parts of the subject matter which are fundamental and to which the pupils have directed the most attention. Nothing should receive attention that is not worth perpetuating in the course. This means that every test should contain a thorough sampling of the fundamental ideas of a topic or a course for the complete mastery of which the pupil is held responsible. In other words the test must be comprehensive. This has never been true of the traditional "essay type" examination.

If the two preceding principles are adopted, the test will be ranked as valid or as having "validity"—the property of a test which is supposed to represent the degree to which a test measures what it is intended to measure.

3. The scoring of every test should be made as objective as possible so that different teachers may obtain exactly the same results. This means that the personal factor must be largely removed from the scoring of a test, and that teachers in making up tests should be more careful about the mechanics of them. If this principle is kept in mind, not only will the response of the pupils be more uniform, but the marking practices of those who score the papers will be less variable. This is done by making the number of items in the test as large as possible and making each of these items have a definite response to which all persons scoring the test papers will readily agree.

The failure to do this is the glaring error of many "essay type" examinations.

4. Every test should be reliable. Reliability is that property of a test which measures the degree to which a test "measures what it really does measure." For, a test may measure what it is intended to measure, but it may measure it very unreliably. The determination of the reliability of a test is, however, a mere technical matter and need not be further discussed here. It is discussed fully in some of the newer books on examinations.*

5. Every test should be so constructed that it is almost self-administering and so that it can be easily given and scored by any intelligent person who may or may not have had much mathematical training. In the past this has not often been done. Teachers more generally need to transfer the enormous amount of time traditionally given to test papers to the matter of preparing them in accordance with the principles here outlined. The motto should be "Hard to make but easy to give and score."

6. Finally, every test should make it possible to set some sort of standard of achievement for a pupil within his own group or against his own record. Of course standards are set from without the group but they should not be considered more important.

Such a procedure as that recommended above will enable us to measure adequately the pupil's knowledge of the subject matter studied by the majority of the class, to assist the teacher in selecting only important topics, to stimulate the pupil to greater effort, and to aid him in self-instruction.

It should be clearly understood that the method of devising and using tests in mathematics is independent of the type of subject matter taught in the course. The emphasis, however, is placed upon the importance of testing what has been taught regardless of the nature of the subject matter.

*G. M. Ruch, Improvement of the Written Examination, Scott, Foresman and Company, 1924.

It is clear that the old type examinations have not been based on a careful consideration of the above principles.

Practice Tests

In any testing program the teacher should introduce practice tests both as diagnostic instruments and as teaching devices. In fact such tests are so important that they are coming to be an integral part of the newer text books. They serve a purpose in testing specific objectives which have to be covered in a very short period of time—a function unknown to most standardized tests. Whether such tests should be timed is a matter for the teacher to determine. Certainly no undue emphasis should be placed upon the matter of speed at the expense of accuracy. A business man need not be particularly rapid in his computations, but there must be no question as to his accuracy. An example of such a test is shown on pages 125 and 126.

It is clear that at best much time has been lost in trying to combine mass instruction and individual instruction in some economical fashion. One thing we seem to have learned, namely, that of the two, individual instruction should receive the greater emphasis. Professor Kilpatrick says that "a child learns the responses which he makes." The truth of this statement makes the use of practice tests imperative. It is here that the value of practice tests appears in enabling the teacher to discover quickly the pupils who are in need of help, and to keep from wearying the brilliant pupils with work that they do not need. Furthermore, they help the pupil to measure his own progress in any given topic, and to be more intelligent in calling upon the teacher for assistance when this progress is not satisfactory. There is little danger that such tests will be overdone if they are also used for diagnostic purposes.

For the pupils who need to have certain skills developed further practice tests are invaluable. For those who do not need it they enable the teacher to excuse such pupils from

further practice before they are disgusted with the subject.* Overlearning is better than underlearning, but why should drill upon anything be continued beyond the stage of diminishing returns? If only we can be a little more scientific in our teaching, we shall save hours of time which can be devoted to additional topics in mathematics. This will be of great interest and value to many pupils who at present are compelled to drill for days upon material which they know perfectly well and which as a result becomes an intolerable bore. This is particularly true of gifted pupils who, in many respects, as already stated are often the most retarded pupils in the entire school system.

TIMED PRACTICE TEST

Time, 5 min.

Write the answer to each problem, but do not solve any equation:

1. What is the cost of 8 oranges at c cents each?

2. A man had m dollars and lost n dollars. How many dollars had he left?

3. If a boy is n years old, how old will he be x years from now?

4. Three times a certain number decreased by 4 is 20. Find the number.

5. Find the number of feet of wire needed to inclose a lot l feet long and w feet wide.

6. If 4 is subtracted from three times a certain number and 8 is added to twice the number, the results are equal. Find the number.

7. Find two consecutive numbers whose sum is 47.

8. Find two consecutive odd numbers who sum is 76.

9. One man has three times as much money as another man. If they both together have $5400, how much money has each?

*W. A. McCall, How to Measure in Education, The Macmillan Company, 1922, p. 118.

10. A rectangular garden plot is 8 ft. longer than it is wide. Its perimeter is 332 ft. What are the dimensions?

11. The sum of three numbers is 392. The second is five times the first and the third is eight times the first. Find the numbers.

12. A rope 90 ft. long is cut into two parts so that the longer part is three times as long as the shorter part. Find the length of each part.

13. Eight times a number decreased by one fourth of itself is 124. Find the number.

The median number of rights on this test is 9.

To determine the median number of "rights" (exercises correctly solved) in any class, find the score (the number of rights) which has as many above as below it. It is not expected that many, if any, pupils in an ordinary class will complete all the exercises correctly in the time allotted for any test.

The method of determining the time on such tests is to stop the work as soon as two or three of the pupils have finished. In this way the test gives a measure of each pupil in a class. A stop watch should be used, so that the results obtained will not vary greatly from class to class. The attempt was made in this test to get the time factor as nearly correct as possible, but it is probable that this will have to be further adjusted if the test is given to a larger number of pupils than the three hundred that were available in this experiment.

The pupils were not asked to solve the equations in any of the exercises above because this test was thought of as a measuring device rather than a teaching device. In the latter case the pupils would have been asked to solve each equation.

At frequent intervals it will be necessary to construct composite tests which contain certain representative exercises from the practice tests that have preceded. These tests like practice exercises are also coming to be an integral part of modern textbooks and the more scientifically and carefully such tests are worked out in textbooks the better the results should be. The method of procedure in making such tests can

best be understood by taking some definite objectives upon which it will be assumed we are agreed and building up the practice tests and composite tests which seem to be necessary to ascertain to what extent our aims are being realized.

The Testing of Specific Objectives

Let us assume for the sake of illustration that we have decided upon the following specific objectives* in teaching the formula:

1. To develop certain rules of mathematics and to translate them into formulas.

This means that pupils should understand the meaning of the formula as a shorthand rule of mathematics. This rule should grow out of their experience if possible. At any rate they should be told what the formula means as far as possible. Here is where algebra begins.

2. To translate certain formulas into rules of mathematics.

This means that pupils must know how to use a formula when the need arises. That is, in getting certain required results they must be taught how to decide which is the proper formula to use.

3. To evaluate certain formulas; that is, to find the values of certain letters when the values of the others are known.

These formulas should be of a difficulty no greater than that found in the operations which the pupils have been taught or which they may be expected to understand.

4. To derive one formula from another.

This means that the pupil must be able to solve a formula for one letter in terms of the other letters in that formula. This involves the ability to solve equations by means of which the "subject of the formula" is changed.

5. To represent by a graph certain formulas of a type no more difficult than $F = 9/5\ C + 32$.

This involves the ability to make a table of values of a formula.

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*W. D. Reeve, "Objectives in the Teaching of Mathematics," *The Mathematics Teacher*, November, 1925, Vol. XVIII, pp. 6-7.

6. To understand the idea of the dependence of one quantity upon another.

This involves the ability to appreciate the idea of one variable as a function of another.

The type of equations involved are as follows:

(1). $2w=6$. (2). $\frac{1}{2}h=9$.

(3). $p+5=8$. (4). $n-4=7$.

The solution of such equations implies a knowledge of how to use the four fundamental operations.

We may then consider that the results of the tests on pages 134 and 136 inclusive will tell us whether we are realizing these objectives. It is assumed of course that some teaching has taken place and that practice tests like those discussed have been given.

Experimental Work With Tests

In order to illustrate further how tests should be used to check up on the more minute details of our teaching I will give the results of a recent experiment of my own in teaching an eighth grade class in the junior high school how to subtract one directed number from another. I chose subtraction because it is probably the most difficult of the four fundamental operations. Since in most courses pupils are asked to subtract horizontally as well as vertically I made up two tests and gave them two days after the teaching of subtraction was begun. The best pupils finished the vertical subtraction test in $2\frac{1}{2}$ minutes and the horizontal subtraction test in 5 minutes. If this is characteristic of what would happen generally it is clear that vertical subtraction is far more economical and might well be the way to teach subtraction if it were not for the fact that in collecting terms in an equation it is necessary to deal with the terms horizontally.

In each test the first thirty-three examples represent the different types of difficulty and the rest of the test was added merely to keep the brighter pupils busy till the slower ones had finished the first thirty-three. It is interesting to note that

after two days teaching several pupils were able to obtain 100% mastery on each test. These tests are on pages 132-133.

The next question was to ascertain whether after teaching the subtraction of one directed number from another I might expect the pupils to subtract one algebraic monomial from another without teaching the specific operation. In order to find out the answer to this question I arranged two tests which involved new difficulties not included in the two preceding tests where no letters were involved. Instead of having the adding and subtracting of directed numbers purely, I introduced monomials of a simpler type. Some pupils, however, were able to get all of the exercises correct without any additional teaching but this was not true of all by any means. The results, however, indicate that for some pupils we can rely on transfer to take care of the situation. In other cases it is clear that if we want transfer to occur, we must set out to obtain it.

What I have done for algebraic subtraction I have proceeded to do in developing other skills and abilities, namely, to analyze each topic so that I have in each test all the possibilities for error which the pupil may make, to try these out on the pupils to see which ones have complete mastery and which ones need further drill or remedial instruction. For a more detailed discussion of some more of the interesting findings of such work the reader is referred to my fuller discussion of the teaching problems in high-school mathematics.*

We need next to devise some kind of machinery for releasing the more brilliant or successful pupils from further drill on a topic, at least temporarily, when they have complete mastery, and for bringing the slower or duller pupils up to a reasonable standard of mastery before they are permitted to proceed.

Such a procedure as I have outlined above will bring out two very important points. They are not new, and I merely give them here for emphasis. First, it will be discovered that

*W. D. Reeve, *A Diagnostic Study of the Teaching Problems in High School Mathematics*, Ginn and Company, 1926.

many things which the teacher has considered easy are really difficult for all pupils; and second, that some things which the teacher thinks difficult prove to be relatively easy for all pupils. This shows the enormous advantage of diagnostic practice tests which have in them every possible chance for error which a child may make in a given topic. It is only in this way that difficulties in learning can be discovered and overcome so that progress can be made. Naturally enough such tests as those I have just described must be repeated from time to time if the pupil is to be expected to do them successfully. The teacher often is not justified in saying that a pupil has not been taught a thing merely because he does not show a knowledge of it on a certain test. This is one of the major faults of "extramural" examinations. They often seem to test what the pupil has learned when in reality they do not. The pupil may really have learned algebraic subtraction well enough for a time, but in the mean time he may have forgotten certain details through lack of practice. It is time for teachers in one part of the educational system to stop condemning those in another part because their pupils do not know what is expected of them. Such teachers first need to make sure what the possible difficulties are in a certain topic. Second, they need to test new pupils who come to them and if these pupils do not know all the necessary things they should give them the chance to relearn or, if necessary, to learn for the first time the fundamental things necessary in making the proper advance.

Our present method is often to set a test which overlooks most of the items mentioned above and then condemn the pupils and all their previous teachers for all the errors that appear on such a test.

"New-Type" Objective Tests and Their Significance

Since everyone no doubt will agree that the traditional "essay type" of written examination is not adequate to meet our modern standards and since the standardized tests or scales of recent years have proved unsatisfactory in many ways, let us turn our attention briefly to some of the "new-

type examinations" which are coming into favor and into wider use and then see to what extent they can be utilized both as diagnostic instruments and as measures of achievement. Although in their use we may sacrifice an important function of the traditional examination, they serve other interests which are important and which in the long run are not served by the ordinary examination or the standardized test. This does not mean that the "new-type examination" should replace the ordinary written examination altogether any more than the standardized test should replace any such examination, but it does mean that the "new-type examination" will reinforce and supplement the older type of examination and give us results which we would not be able to get without its use.

Certainly the "new-type examination" will prove superior to standardized tests in two respects. It will enable us to cover a greater range of skills and abilities in the same length of time and it will also be more easily made into a teaching device which can be used for both instructional and drill purposes.

It has been charged that if the "new-type examinations" are used they will degenerate into tests of pure memory work, but this is not true. It will be true only when the one who makes the test is careless and such carelessness is common to the traditional type of test. Some of the new type tests lend themselves admirably to thought provoking questions and the more care that is exerted in making them, the better they can be made.

Space will not permit me to go into a long discussion of the pros and cons of "new-type" objective tests. Doubtless there are some rough spots to be smoothed down before we can say that the newer types of tests are what they should be. The "true-false" test is at present a source of much discussion. No attempt has here been made to justify any of these newer tests. The reader is encouraged to experiment with them in a serious manner to see if our combined efforts cannot bring about a set of tests which shall be valuable.

The following tests are illustrations of the type of tests the present writer has discussed.

VERTICAL SUBTRACTION OF DIRECTED NUMBERS

Perform each of the following subtractions, writing the result below the line:

+8	+3	0	+2	5	3
+6	−7	+7	2	+3	+5
+8	−4	+6	0	+4	+3
−6	+8	0	−4	2	+3
−8	6	+4	−6	+2	+4
+5	8	+4	0	+5	−4
9	3	+5	+7	+3	−5
7	−6	−5	+4	−6	+5
8	−4	−6	+6	−2	6
−6	+7	+6	−4	+4	6
−9	−5	3	−5	3	4
7	−8	3	+2	5	−4
−8	+5	4	4	3	−7
−5	8	−4	2	−5	7
9	4	−5	6	−2	−8
+7	+6	5	−4	7	−8
+8	0	−7	−7	−2	+3
6	5	−7	4	−5	3
+5	4	8	−6	+2	2
+8	0	+8	−4	5	+2

HORIZONTAL SUBTRACTION OF DIRECTED NUMBERS

Perform each of the following subtractions, writing the result after the sign of equality:

1. $+9+(+7)=$	21. $0+(+9)=$	41. $0+4=$
2. $+9+(-7)=$	22. $+9+0=$	42. $4+0=$
3. $-9+(+7)=$	23. $+6+(+6)=$	43. $+2+(+5)=$
4. $9+7=$	24. $+6+(-6)=$	44. $+2+(-5)=$
5. $9+(-7)=$	25. $-6+(+6)=$	45. $-2+(+5)=$
6. $-9+7=$	26. $6+6=$	46. $2+5=$
7. $-9+(-7)=$	27. $6+(-6)=$	47. $2+(-5)=$
8. $9+(+7)=$	28. $-6+6=$	48. $-2+5=$
9. $+9+7=$	29. $-6+(-6)=$	49. $-2+(-5)=$
10. $+5+(+8)=$	30. $6+(+6)=$	50. $0+(+5)=$
11. $+5+(-8)=$	31. $++6=$	51. $+5+0=$
12. $-5+(+8)=$	32. $0+6=$	52. $+3+(+3)=$
13. $5+8=$	33. $-6+0=$	53. $+3+(-3)=$
14. $5+(-8)=$	34. $+6+(+4)=$	54. $-3+(+3)=$
15. $-5+8=$	35. $+6+(-4)=$	55. $3+3=$
16. $-5+(-8)=$	36. $-6+(+4)=$	56. $3+(-3)=$
17. $+5+8=$	37. $6+4=$	57. $-3+3=$
18. $5+(+8)=$	38. $6+(-4)=$	58. $-3+(-3)=$
19. $0+9=$	39. $-6+4=$	59. $0+3=$
20. $9+0=$	40. $-6+(-4)=$	60. $-3+0=$

EVALUATING FORMULAS

Given the formula $A=lw$, which means that the area of a rectangle is the product of the length and width, complete each of the following:

1. If $l=4$ and $w=3$, I find the product of 4 and ..... and thus know that $A=$
2. If $l=3$ and $w=4$, I ..... 3 by 4 and find that $A=$
3. If $l=6$ and $w=3$, I find by multiplication that $A=$
4. If $l=3\ 1/2$ and $w=6$, I multiply ..... by ..... and find that $A=$
5. If both l and w have the value 9, then A=
6. If $l=8$ and $w=4\ 1/4$, I multiply mentally and find that $A=$
7. If $l=4\ 1/2$ and $w=3\ 1/2$, I multiply as shown below and find that $A=$
8. If $l=5\ 1/4$ and $w=3\ 1/2$, I multiply as shown below and find that $A=$
9. If $l=6\ 3/4$ and $w=2\ 7/8$, I multiply as shown below and find that $A=$
10. If $l=3.6$ and $w=4.7$, I multiply as shown below and find that $A=$

INFERENCES ON PERPENDICULAR LINES

If a statement in the following list is true, underline the word "true," if it is false or partially so, underline the word "false."

1. Perpendicular lines will never meet no matter how far they are produced. *true—false*
2. Perpendicular lines make an angle of 0 degrees with each other. *true—false*
3. Perpendicular lines are always the same distance apart. *true—false*
4. A line parallel to one of two perpendicular lines is parallel to the other also. *true—false*
5. A line perpendicular to one of two perpendicular lines is perpendicular to the other also. *true—false*
6. Two perpendicular lines make an angle of 90 degrees with each other. *true—false*
7. If a line is parallel to one of two perpendicular lines, it is perpendicular to the other. *true—false*
8. If a line is perpendicular to one of two perpendicular lines, it is parallel to the other. *true—false*
9. If two perpendicular lines meet, the four angles they form will be right angles. *true—false*
10. The line which measures the shortest distance from one of two perpendicular lines to the other is parallel to both. *true—false*

DEPENDENCE OF QUANTITIES

In each of the blanks in the following statements insert the word which makes the best sense:

1. The cost of a sirloin steak depends upon the ..... per pound.

2. The value of the algebraic expression $5x$—3 depends upon the value of .....

3. The circumference of a ..... depends upon the length of the diameter.

4. The cost of sending a package by parcel post to the third zone depends upon the ..... of the package.

5. Doubling the length of the radius of a circle ..... the circumference.

6. The number of yards of wall-paper border needed to go around a rectangular room depends upon the ..... and ..... of the room.

7. The number of theater tickets that can be bought with a 10-dollar bill depends upon the ..... of each ticket.

8. The ..... that an automobile can travel at an average ..... of 30 mi. per hr. depends upon the ..... taken for the trip.

9. The volume of a circular cylinder depends upon the ..... and the ..... of the base.

10. The time that it takes you to fill in all the blanks on this page at an average rate of 4 blanks per minute depends upon the ..... of blanks.

11. The interest received per year from an investment of $500 depends upon the ..... of interest.

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THE DEVELOPMENT OF MATHEMATICS IN THE JUNIOR HIGH SCHOOL

By William Betz

INTRODUCTION

The following report comprises, first, a brief account of the junior high school as an institution, and second, a survey of the present trend in junior high school mathematics. The prominence given to the first part of this report is due to the belief that an intelligent grasp of the factors which are molding the character of any subject in the curriculum is impossible without continuous attention to the educational situation in its entirety. Naturally, an exhaustive treatment of the many questions suggested by this investigation would far transcend the limits of this Yearbook. Hence there is no attempt at completeness at any point. Moreover, for obvious reasons, material which may be supposed to be familiar to the majority of progressive teachers has been either omitted or presented in outline form. In the interest of a broader point of view, numerous quotations and references were introduced throughout the discussion.

I. THE JUNIOR HIGH SCHOOL AS AN INSTITUTION

1. *Definition of the Junior High School.* The term, "junior high school," has been in general use about fifteen years. Nevertheless, there is no perfect agreement as to its exact meaning. Many definitions have been formulated. None seems entirely adequate. From a mere state of mind, a "feeling" or an "idea" without corporeal existence, the junior high school has developed into an institution of commanding importance. At first, a single sentence served to describe that

idea. Its growing complexity at present seems to require numerous paragraphs.

Thus, in 1919, the North Central Association Commission on Secondary Schools adopted the following definition:

> A junior high school is a school in which the seventh, eighth, and ninth grades are segregated in a building (or portion of a building) by themselves, possess an organization and administration of their own that is distinct from the grades above and the grades below, and are taught by a separate corps of teachers. Such schools, to fall within the classification of junior high schools, must likewise be characterized by the following: 1. A program of studies decidedly greater in scope and richness of content than that of the traditional elementary school. 2. Some pupil choice of studies, elected under supervision. 3. Departmental teaching. 4. Promotion by subject. 5. Provision for testing out individual aptitudes in academic, prevocational, and vocational work. 6. Some recognition of the peculiar needs of the retarded pupil of adolescent age, as well as special consideration of the supernormal. 7. Some recognition of the plan of supervised study.

After listing forty-four aspects of the junior high school movement, Briggs* invites the reader to make his own definition. More recently, Davis† finds that a junior high school worthy of the name has as many as seventeen different characteristics. In his opinion, "the junior high school may be defined as a school unit developed in the United States within recent years and designed to furnish to all pupils, between the ages of twelve and fifteen years approximately, (1) continued common education on high elementary levels, and (2) the beginnings of a differentiated or secondary education adapted to each pupil's individual needs."

2. *Historical Background of the Junior High School.* Students of the junior high school movement have shown that it is the result of an older and much more comprehensive movement which may be traced far back into the past. The essential facts concerning its more recent development have been summarized by Clement‡ as follows:

> The first vigorous, conscious discussion concerning the reorganization of the eight-four plan occurred during the last decade of the nineteenth century. President Eliot in 1888 strongly advocated reorganization. The Committee of Ten in 1892-1893, and the Committee of Fifteen in

*Briggs, The Junior High School, p. 56.
†Davis, C. O., Junior High Education, Chap. I.
‡Clement, J. A., Curriculum Making in Secondary Schools, p. 201.

1895 also considered this problem. During the first decade or more of the twentieth century the question of reorganized secondary education was re-discussed, directly and indirectly, by the Committee on the Articulation of High Schools and Colleges, in 1911; by the Commission on the Reorganization of Secondary Education, in 1913-1914; by the Committee on Economy of Time, in 1913; by the Department of Superintendence, in 1916; and by the North Central Association, from 1918 to 1922, in its yearly conferences.

The history of the contributions, by individuals and committees, which led to the junior high school movement, has been formulated by Bunker,† Bennett,‡ Briggs,§ Davis‖ and others.¶

It appears that after numerous pioneer efforts in various parts of the country, "the real beginning of the present junior high school or intermediate school movement is probably to be found in the reorganization of the school systems in Columbus, Ohio (1908), Berkeley, California (1910), Concord, New Hampshire (1910), and Los Angeles, California (1911)." Since that time it has spread rapidly, and is now extending in all parts of the country.

The following table, due to Briggs, may be of interest in this connection:

THE YEARS IN WHICH 272 JUNIOR HIGH SCHOOLS WERE ESTABLISHED

Year	*Number*	*Year*	*Number*
Before 1900	2	1912	21
1905	1	1913	27
1907	1	1914	44
1908	3	1915	76
1909	3	1916	68
1910	11	1917	6
1911	9		

†Bunker, Frank F., Reorganization of the Public School System. United States Bureau of Education, Bulletin 1916, No. 8.

‡Bennett, G. Vernon, The Junior High School, Chap. II.

§Briggs, Chapter II.

‖Davis, Chapter II.

¶Valuable summaries of many articles and reports concerning this period will be found in Uhl, W. L., "Principles of Secondary Education," Silver, Burdett and Company, 1925.

3. *The Extent of the Junior High School Movement.* In 1916, according to reports received by Douglas,* there were between three and four hundred junior high schools or intermediate schools. A careful estimate by the Bureau of Education† in 1918 indicated the existence of 557 junior high schools. In 1922 Inglis‡ estimated the total number at more than one thousand, while Clement (1923)§ speaks of the "fifteen hundred or more junior high schools of the country."

At the present time, according to Koos,‖

> thousands of other communities are giving serious consideration to the proposal to effect immediate or early reorganization. Millions are being voted by some of these cities for buildings for properly housing the new institution. Junior high school textbooks in a number of subjects have long been on the market. Hardly an educational convention meets which does not give the discussion of the problems of this new school a prominent place on its programs. Educational periodicals devote much space to articles on the junior high school. Departments of education in colleges and universities are offering courses concerned exclusively with its problems and these and other training institutions claim to be preparing teachers for it. State legislatures are enacting laws to authorize its establishment or to regulate its operation. These are some of the evidences that the junior high school is now one of our most engrossing educational concerns.

4. *The Changing Character of the Junior High School Idea.* It is very important to remember that the junior high school idea has been undergoing constant revision. Davis,¶ in an admirable summary, calls attention to the three principal periods or phases which the junior high school movement has passed through up to the present time. Whether viewed from the standpoint of aims, of methods, or of content, each decade since 1890 seems to have faced the problem in a different manner. Thus, during the first period, from 1890 to 1900, the movement was guided and influenced largely by university administrators; in the second period, from 1900 to 1910, by public school authorities; in the third period, from 1910 to the present time, by professional students of pedagogy.

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*Douglas, A. A., The Junior High School, p. 27.
†Statistics of Public High School, Bureau of Education Bulletin, 1920, No. 19, pp. 111-112.
‡Inglis (Kandel), p. 263.
§Clement, p. 204.
‖Koos, pp. 9-10.
¶Davis, pp. 28 and 29.

Briggs† reminds us that

the reorganization of schools on the 6-6, 6-3-3, or 6-2-4 plans was not always due primarily to a conception of definite programs for educational reforms. In some instances a superintendent had an outgrown high school building which was too good to destroy and yet not suited for all the elementary grades; in others there was a growth of population in a section of the city remote from the existing high school; in others still there was overcrowding that could best be relieved by a building in which pupils of the upper grades and the first year of the high school could be congregated.

Inglis‡ describes the change that has taken place in the aims of the junior high school, as follows:

The *earlier* proposals for reorganization were concerned primarily with plans for an earlier beginning and a longer period of secondary education of the sort already represented in the high school, and the proposers were actuated in large part by ideas of a *longer period of preparation* for college. *Later proposals*, however, and those which have been effective in developing the junior high schools, were in keeping with our changed conceptions of the functions of secondary education and emphasized a *different kind of education* rather than a downward extension of the secondary education previously provided.

Unless these changing conceptions are constantly kept in mind, "there is the greatest danger that the real reorganization, necessary, and intended, may be lost sight of in the reorganization of administrative divisions."

5. *The Purpose of the Junior High School.* Although Briggs* obtained from 266 junior high schools specific reasons for their establishment, it is the contention of Clement that the real motives which led to the creation of the junior high schools as a distinct section of the public school system are already being forgotten. Davis‖ thinks that "of all the functions of the junior high school, that which seeks to aid pupils in discovering their own capacities and limitations, interests, and distastes, powers and weaknesses, is the most important. It is this function, above all others, that justifies the reorganization of schools on a new basis."

The relative prominence attached by school documents or

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†Briggs, p. 33.

‡Inglis, Alexander J., Secondary Education, Chap. X (p. 263) of Kandel, I. L. (Editor), Twenty-five Years of American Education, The Macmillan Company, New York, 1924.

*Briggs, p. 34.

‖Davis, p. 99.

educational leaders to such peculiar functions as retention of pupils, economy of time, recognition of individual differences, guidance, etc., was studied by Koos.‡ Extremely significant, if not depressing, is the low rating given to the aim which stresses "better scholarship."

Acording to Inglis,§ the main objectives of secondary education are (1) the social-civic; (2) the economic-vocational; and (3) the individualistic-avocational. To realize these objectives, he states that the school should exercise the following six functions:

The integrating function.	The diagnostic function.
The differentiating function.	The propaedeutic function.
The selective function.	The adjustive function.

Each of these functions should operate in the junior high school as well as in the senior high school. The integrating function should tend to foster social solidarity; the differentiating function should aid in developing personality; the selective function should sift out the more fit from the less fit; the adjustive function should empower individuals to relate themselves to the "everchanging demands of dynamic society"; the propaedeutic function should prepare the pupil to continue his studies into their more advanced stages; and finally, the diagnostic function should help the pupil to discover his own elements of strength and weakness and to plan his life career accordingly.‖

6. *Types of Organization.* As might have been expected, there is no uniformity in the organization of the junior high school. A dozen or more varieties of its external organization may be found in various sections of the United States. This fact leads Koos to say that "the junior high school is hardly the same thing in any two communities." The forms found most frequently are the 6-2-4, the 6-3-3, and the 6-6 plans. Hines,* who in 1917 reported nine different types in the 300 cities included in his investigation, states that the 6-3-3 plan has now become the most popular. This claim is supported by a study made by Superintendent Pratt† in 1922, in which it is stated that 21 out of 26 cities of 100,000 population or above have adopted that scheme of reorganization.

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‡Koos, L. V., The Junior High School, Chap. II, pp. 18 and 19.
§Inglis, A. L., Principles of Secondary Education, Chap. X.
‖As summarized by Davis, p. 102.
*Hines, H. C., "Junior High School Curricula," p. 2.
†Pratt, O. C., "Status of the Junior High School in Larger Cities," School Review, November 1922.

The struggle between the 6-3-3 plan and the 6-6 plan is of particular interest to the "upper high school". An extensive literature has arisen on the relative merits of the two types.‡

It is probable that eventually each community will select that form of organization which best fits its local needs.

7. *The Program of Studies.* Many questions pertaining to the program of studies in the junior high school are far from settled. In particular, there is a continuous struggle between enrichment and economy of time, between uniformity and variation, between "terminal" and "preparatory" values. Nevertheless, a clearer view of the outstanding problems seems to be emerging.

Thus, James M. Glass, on behalf of the Pennsylvania State Department of Public Instruction, in 1922, stated the case as follows:

> It is the difficult mission of the junior high school to continue a program of studies carried through the six years of the elementary school, modify and enlarge this program for the realization of its own purposes, and in turn prepare for advanced types of curricula in the senior high school. It is plain that this can be done only through successive periods in the transitional process. Briefly, these periods are four in number: A. Adjustment (Low Seventh). B. Exploration and Pre-view (High Seventh and Low Eighth). C. Provisional Choice of Electives (High Eighth). D. Stimulation (Ninth Year).

The dangers of indiscriminate expansion are stated forcefully by a number of writers.* As to prevailing types of curriculum organization, Koos† has investigated the relative prominence of the following: 1) the single-curriculum type,

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‡See especially, Sachs, J., "The American Secondary School," 1912, pp. 108-110;
Wheeler, George, "The Six-year High School," School Review, April 1913;
Bagley, W. C., "The Six-Six Plan," School and Home Education, September 1914, November 1914, and March 1915;
Harris, J. H., "The Six-Six Plan," Journal of Education, Vol. 91, 1919, pp. 535 et seq.;
Shaefer, Harry M., "The Six-Year Unified High School versus the Senior and Junior High School," Proc. N. E. A., 1920, pp. 221-226;
Davis, C. O., "Junior High School Education," Chap. V.

*See, especially, Davis, Chap. VII; Briggs, Chap. VI; Clement, Chap. X; Inglis, Chap. XI, pp. 412-418.

†Koos, Chap. IV.

2) the multiple-curriculum type, 3) the "constants-with-variables" type. He finds the second type in use in more than a fourth of a large number of unselected junior high schools, while the third is to be found in an increasing proportion of schools.

In any case, it is becoming clear that "no single program of studies can be formulated that will meet *all* conditions. Programs of studies must grow out of local as well as national and universal needs. It is as absurd to draft a program of studies for schools in general as it is to prescribe a given medicine for diseases in general. Each school district has special problems, interests, and limitations which must be taken into account."

8. *Curriculum Problems.* An examination of hundreds of printed courses of study has led competent critics to observe that thus far only a beginning has been made at accomplishing desired ends.* Many so-called junior high school programs of study and curricula "have neither been re-cast nor re-administered in any way different from that of the perfunctory practice under the traditional eight-four regime." It should be emphasized, of course, that "the whole problem is not so much one of new courses (curricula) or new administrative machinery, as it is of reorganization and redirection of much of the secondary school work in terms of twentieth century social needs and values."

The crying need of the hour is conceded to be a *scientific* curriculum policy.† This means that subjective opinion, however valuable, can no longer be the exclusive basis of curriculum readjustments. There must also be scientific inventories and analyses of the individual and social interests and needs of pupils. Moreover, "it is impossible to formulate and outline secondary school curricula apart from consciously

*See, especially, James M. Glass, "Curriculum Practices in the Junior High School and Grades 5 and 6," University of Chicago Educational Monograph, No. 25, November 1924.

†See Clement, Chap. XII; Inglis, Chap. IX; Uhl, Part VI (pp. 585-662); Briggs, Thomas H., The School Review, February 1923, pp. 109-115.

recognized objectives." Finally, there must be a continuous process of re-definition, because modern society is never static.

Hence, in planning secondary school curricula, "one of the *first* steps is a clear statement of objectives in terms of the growing needs of pupils, and of the society in which they live. A *second* important administrative step is the provision for an *adequate time allotment* for the realization of such objectives as are definitely outlined."

Viewed in the light of these statements, it is clear that "curriculum making in our secondary schools at present is no mean task. It is a complicated project, a stupendous undertaking in the midst of the complexity of problems arising out of the current changing social order. Obviously, no single individual can hope to suggest *the final procedure.*"‡

9. *The Development of Method.* During the past thirty years an almost revolutionary change has taken place in the attitude of leading educational thinkers toward the problem of method.* This change has been characterized as a transition from Herbartian formalism and receptivity to the functional program inaugurated and sponsored by Dewey, Thorndike, Kilpatrick, and their numerous disciples. The old faculty psychology has been replaced by the mechanistic stimulus-response hypothesis. A new doctrine has arisen with reference to the transfer of training. This doctrine, instead of *"exploding the myth of mental discipline,"* no longer questions the *fact* of transfer, but is trying to determine its extent and its modus operandi by scientific methods. Finally, the use of statistical methods, as applied in the field of experimental pedagogy and in the measurement movement, is profoundly modifying classroom technique. Naturally, these outstanding

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‡Attention may also be called at this point to the publications of well-known "curriculum builders," e. g., Bonser, Charters, Bobbitt.

*For valuable reviews of this period, see Maddox, William A., "Development of Method," Chap. VI, of Kandel (Editor), Twenty-five Years of American Education; also Hillegas, Milo B., "The Problem of Method," United States, pp. 557-596, Educational Yearbook (1924), International Institute of Teachers College, Macmillan Company, 1925.

transforming influences have begun to affect the pedagogic procedure of the best junior high schools.

Only five of the many consequences of this new orientation can be mentioned at this point: 1. The project-problem method.† 2. The conscious application of tested objectives. 3. Supervised study. 4. Experimental teaching on a nation-wide basis. 5. The use of educational tests for purposes of classification and diagnosis, and for measuring progress.

An enormous literature has arisen with reference to each of these five factors. The newer contributions seem to warrant the feeling that at last we are beginning to develop a more discerning and critical judgment which refuses to be influenced by every new panacea.‡ Above all, the feeling is growing that "any move in the direction of the establishment of a *general method* is necessarily *fallacious*. The way to learn to spell a word is *not* the way to learn to play on the piano. Even the motivation is not the same in any but a rhetorical sense of the word."

10. *The Preparation of Teachers.* In 1918 the North Central Association of Colleges and Secondary Schools unanimously adopted the recommendation

> that the standard of preparation for the teacher of the ninth grade of the junior high school be the same as the standard now admin-

†For a critical review of the project method, the reader is referred to Bagley, Teachers College Record, September 1921; Maddox (Kandel), pp. 165-176; Horn, Educational Review, January 1922; Branom, M. E., "The Project Method in Education," 1919.

‡See Dr. Thomas J. McCormack's "Critique of the Measurement Movement," in School and Society, June 24, 1922.—Of course, no one desires to ignore or minimize the epoch-making contributions of the measurement movement. There is, however, a growing resentment of the incredibly naive, and often blundering, encroachment of soulless "educational machinists." Thinking teachers refuse to ignore the "fallacy of the average," or to disregard the appalling danger of "standardized thinking" in a democracy which depends for its very existence on intelligent initiative and superior leadership. Such sentiments seem to be fairly universal. From Germany, the home of experimental psychology, comes the message that "on the whole, practical teachers are not enthusiastic over intelligence tests; they would like to have, in addition, a *sound* method for measuring in the fields of emotion and will." (Educational Yearbook, 1924, p. 316. See also, Lehmann, R., Das doppelte Ziel der Erziehung, Berlin, 1925, pp. 60-78.)

istered for secondary teachers by the North Central Association. An equally high standard of preparation for the teacher of the seventh and eighth grades of the junior high school should be insisted upon as soon as practicable.

Naturally, as one writer remarks, "these standards represent *ideals*." He goes on to say that "whatever standards are set up, provided they be not mandatory, a superintendent is likely to select the best teachers that he can find, most probably in his own system, regardless of academic training or degrees."

A number of studies have been made with reference to the preparation and training of junior high school teachers. Thus, Smith,* in 1922, after examining the catalogues of training institutions, found that practically one-third of the states support some activities in the training of teachers and supervisors for junior high schools. There were 125 methods courses in 48 institutions. The average experience of 473 teachers in 120 schools was reported by Evenden† to be 7.5 years. The plan followed at Rochester, New York, in the selection and the training of a group of teachers for the first junior high school of that city has been described by Superintendent H. S. Weet.‡

In general, the opinion seems to prevail that teachers recruited from the grammar schools excel in classroom control and in teaching technique, while college graduates with specialized training are superior in scholarship. The unsolved problem is a unison of method and content.

11. *Appraisal and Outlook.* Undoubtedly it is still too early to expect anything but a provisional estimate of the accomplishments and the standards of the junior high school.‖

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*Smith, H. J., "Special Preparation for Junior High School Service," Educational Administration and Supervision, December, 1922; see also Gosling, "The Selection and Training of Teachers for Junior High Schools," Part I, Eighteenth Yearbook of National Society for the Study of Education.

†Teachers' Salaries and Salary Schedules in the United States, 1918-19, p. 62.

‡Proc. N. E. A., 1916, pp. 1036-42; School Review, February 1915; Educational Administration and Supervision, September 1916.

‖Davis, Chap. XXI and XXII.

In a very real sense, the junior high school is an *evolving* institution, called into existence by many factors, most of them of a slowly maturing force. And it is still being transformed by numerous influences and agencies which are often in serious conflict with each other. It has become the battle ground of a corps of specialists, each expecting the exclusive recognition of his particular point of view. The administrative expert of the junior high school tries to solve its perplexing organization problems by prescribing "broadening and finding" courses, arranging extra-curricular activities, and refining time schedules to an ever increasing extent; the educational theorist wishes to control its methods and classroom policies; educational psychology demands due regard for laws of learning and for individual differences; curriculum experts offer their criteria of selection, evaluation and arrangement; and so on, ad infinitum.†

Under these circumstances, is it astonishing that the finite brain of the mere teacher should have attacks of dizziness? And even the texbook writer, formerly so complacent and so painfully stereotyped, is showing slight symptoms of a beneficent inferiority complex. In fact, multiple authorship has become almost a necessity. That plan, however, may soon be replaced by the coöperative effort of an endowed syndicate of experts. Particularly significant is the nation-wide mobilization of thought on curriculum construction now being directed by a committee appointed by the Department of Superintendence of the National Education Association.

And so, the junior high school is at once the hope and the despair of forward-looking educators. It is the groping attempt of a developing democracy to provide a more adequate educational atmosphere for all the adolescent children of all its people.

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†An amusing and realistic description of the "bunglesome and awkward" situation often prevailing even in superior junior high schools is given by Bruner, H. B., in "The Junior High School at Work," Teachers College Contributions to Education, No. 177, p. 12, New York, 1925.

II. MATHEMATICS IN THE JUNIOR HIGH SCHOOL

1. *Underlying Causes of the Present Status of Junior High High School Mathematics.* Mathematics in the junior high school is still in the making. Neither its purpose, nor its content, nor its method, can be regarded as settled. The agencies which have molded the junior high school as an institution have naturally affected each of the special subjects included in its program of studies. In particular, the form which mathematics is gradually assuming in the junior high school is the composite result of at least three distinct forces: 1. The reorganization movement of the *elementary* school. 2. The reform movement in *secondary* mathematics during the past thirty years.* 3. The change in the educational background as a whole.

It is the failure to understand and to appreciate at its true value the legitimate and essential contribution of each of these three factors which is causing nearly all the confusion of the present moment. A convincing study of their relative significance would be a grateful task for some able student of education.

2. *How the Content of the New Mathematical Curricula is Being Determined.* According to Professor L. C. Mossman,† leaders in education have viewed the problem of curriculum construction—not always in harmony with each other—in at least seven distinct ways, and with reference to eleven or more different categories. As a matter of fact, a much simpler procedure has been at the bottom of the reorganized curricula in our best schools. Unquestionably, the five principal sources of the new mathematical courses are the following: 1. The subjective opinions of teachers and textbook writers.

*A reliable, connected account of this entire period is urgently needed. Few teachers have access to the original documents. Professor E. H. Moore's famous address is reprinted in this Yearbook. See, also in an earlier chapter of this volume, a review of the work of the important committees, by Professor D. E. Smith who has had such a large share in the progressive movement of the past twenty-five years.

†Mossman, L. C. "*An Analysis of the Theories Basic to Curriculum Construction*," Teachers College Record, May, 1925, pp. 734-739.

2. Committee reports. 3. Objective inventories and analyses. 4. Experimental teaching in representative schools. 5. World opinion.

Obviously, the experience of generations of teachers cannot be ignored. Throughout the period considered the criticisms and suggestions offered by individuals have really been the basis of all subsequent reorganization. Summaries of these suggestions have appeared from time to time. —The reports of the numerous general and special committees which have been at work during the past thirty years have usually given the collective opinion of a small group of *individuals.* Nevertheless, they have profoundly influenced the present trend in mathematics. This is true especially of the Report of the National Committee on Mathematical Requirements.—A summary of twenty-nine objective studies relating to the mathematics of grades 7, 8, and 9, was recently published by Schorling.† These studies are of very unequal value. They include: one analysis of "pupil activities"; six studies of the mathematics needed in general reading; three studies of the mathematics used in business and in social life; six studies of the "academic" uses of mathematics; seven questionnaire investigations. The very common fallacy of depending on the "frequency of occurrence" as a reliable criterion in the selection of subject-matter is strikingly illustrated by a number of these investigations. —Some of the contributions of leading experimental schools are listed by the Report of the National Committee.‡ Moreover, programs for experimental teaching have been suggested by several writers.—Finally, as to world opinion, the reports of the International Commission, and numerous other related publications, have exercised a corrective and broadening influence.

3. *Curriculum Practices in Typical Junior High Schools.* A detailed study of the junior high school curricula in four-

HIGHEST, LOWEST AND AVERAGE NUMBER OF MINUTES ALLOTTED EACH WEEK TO THE DIVISIONS OF REQUIRED GENERAL MATHEMATICS IN FOURTEEN SCHOOL SYSTEMS

	Arithmetic			Algebra			Intiutive Geometry		
Grade	High	Low	Aver.	H	L	A	H	L	A
5	300	125	204	.....	.....	.....	.....	.....	.....
6	275	120	200	.....	.....	.....	.....	.....	.....
7	275	100	192	82	23	23	150	8	90
8	275	75	165	150	69	104	105	9	75
9	225	100	175	300	100	216	125	60	85

†Schorling, Raleigh. "*A Tentative List of Objectives in the Teaching of Junior High School Mathematics,*" Chap. III, Ann Arbor, Mich., 1925.

‡Report by the National Committee on Mathematical Requirements, on "*The Reorganization of Mathematics in Secondary Education,*" 1923, Chap. XII.

teen representative centers was made by Mr. James M. Glass.* Of the extensive statistical material contained in Mr. Glass' report, the following table is perhaps of greatest interest.

Some of the other tabulations, pertaining to the mathematics of grades 7, 8, and 9, may be summarized as follows:

Mathematics of	Time, aver. no. of minutes per week	Time, per cent of total.	Lowest no of minutes per week.	Highest no. of minutes per week.	No. of periods per week.
Grade 7	238.1	14.8%	150	300	4.1
Grade 8	241.4	14.3%	180	300	4.4
Grade 9	298.3	18.3%	200	450	5.2

The principal findings of the report appear to justify these conclusions:

1. There is an enormous discrepancy in the time allotment of these schools.†
2. A glaring disagreement in the content of the courses is noticeable.
3. Real reconstruction of content is far from general. In particular, the function of intuitive geometry is often misunderstood.
4. The fallacious procedure of testing "results", before valid objectives have been formulated, is only too common.
5. The frequent absence of valid objectives and standards makes it only too certain that "until such standards are established by a nation-wide investigation and by committees in each subject field, in which all school systems will have implicit confidence, the present confusion of curriculum practice and the apparently needless waste of school time and public funds must be expected to continue."

4. *Textbooks and Syllabuses.* Thus far, about a score of textbooks have appeared which are devoted either entirely or in part to the mathematical work of grades 7, 8, and 9. They reflect all the conflicting tendencies outlined above. Nevertheless, the total contribution represented by these publications justifies the feeling that the reorganization of mathematics in

*Glass, James M. *"Curriculum Practices in the Junior High School and Grades 5 and 6."* University of Chicago, Supplementary Educational Monographs, No. 25, November, 1924.

†The reader is cautioned against the use of the above data for comparative purposes. The amount of time allowed for home study or supervised study must also be taken into account. This factor varies fully as much as the gross time allowance.

the junior high school has gone beyond the initial stage, and that it compares very favorably with similar efforts in other subjects.

A critical inspection of *all* the available texts shows that much progress has been made in the selection, the motivation, and the arrangement, of the topics included. On the other hand, these texts also seem to warrant the following statements:

1. An excessive variation prevails in the degree of emphasis which is placed on the various ingredients of the curriculum. 2. Intuitive geometry is not receiving its due share of attention. 3. There is no clearly defined policy with reference to purpose or content. 4. With very few notable exceptions, there is no evidence that the materials included were determined experimentally. 5. A majority of the authors evidently lack first-hand classroom experience in genuine junior high schools.

Teachers and textbook "juries" should be reminded that the mathematical curriculum of the junior high school is still in a state of flux, and that, as a result, no single text or syllabus at present can adequately reflect the *ideal* practice of a typical junior high school. All existing books and syllabuses must be regarded as *provisional* and *tentative.* That the National Committee endorsed this point of view very emphatically, is proved by the following quotation from its Report:

The committee is fully aware of the widespread desire on the part of teachers throughout the country for a detailed syllabus by years or half-years which shall give the best order of topics with specific time allotments for each. This desire can not be met at the present time for the simple reason that no one knows what is the best order of topics, nor how much time should be devoted to each in an ideal course. The committee feels that its recommendations should be so formulated as to give every encouragement to further experimentation rather than to restrict the teacher's freedom by a standardized syllabus.

Fortunately, there are indications that in the future the important task of preparing textbooks and syllabuses will be undertaken, to an increasing extent, not by mere theorists, but by capable, well-informed teachers who know from prolonged experience which materials actually function in the lives of children.

5. *General and Specific Objectives of Junior High Mathematics.* The most serious weakness of the present

situation appears to be the absence of generally recognized guiding principles. This confusion of objectives, to be sure, is characteristic of the whole field of secondary mathematics.* A detailed study of the Report of the National Committee, especially of Chapters II and III, would remedy many troubles. Familiarity with the critical reviews of this Report would also be profitable.†

Within the last few years, there has been a growing feeling that teachers need, in addition to *general* criteria of selection, a detailed analysis of the precise scope of each topic. The first comprehensive study of such specific objectives in the field of junior high school mathematics was issued last year by Professor R. Schorling.‡ It includes a basic list of 305 elements (9 attitudes, 63 concepts, 127 abilities, 106 items of information.) The total number of elements considered is 454. The author of this monograph very properly warns his readers against a wrong interpretation of his findings. These statements are particularly significant:

> The technique here employed leaves the door open for growth. It is conceivable that this study repeated ten years hence would result in many and drastic changes. Certainly the same study carried out ten years ago would have resulted in a basic list far different from the one here suggested. . . . . It is not to be expected that this basic list will fit every community, nor win the endorsement of all individuals. On the contrary, the writer does not accept the basic list throughout. . . . . The contention is that this chapter furnishes a good working basis for any group of teachers that sits in conference for the purpose of making a course in mathematics for grades seven, eight and nine for their community.

An extensive investigation of the general and specific objectives of secondary mathematics is being conducted at present by Professor W. D. Reeve of Teachers College.‖ Thorndike's

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*Betz, W. *"The Confusion of Objectives in Secondary Mathematics,"* Mathematics Teacher, December, 1923.

†*"The Junior High School Report."* A Discussion by Walsh, Breslich, Betz, Schorling, Miss Gugle, in the Mathematics Teacher, January 1921. See, especially, Rugg, Harold, *"Curriculum Making: What Shall Constitute the Procedure of National Committees,"* Mathematics Teacher, January, 1924.

‡Schorling, Raleigh, op. cit., Chap. V.

‖The first part of this investigation was published in the *Mathematics Teacher*, November, 1925. It contains a valuable list of references.

contributions in this field have been received with attention.§ Finally, reports of committees, special monographs, and helpful articles in the professional journals,¶ are constantly adding to the fund of available information in this field. In short, if the lack of perspective described above should continue, it will not be due entirely to a dearth of helpful literature.

6. *The Struggle between Stratified and Unified Mathematics.* Just what is the best arrangement of all the items which make up the mathematical curriculum of the junior high school, will always be a debatable issue. The early tendency, found especially in California,* to copy the stereotyped compartment system of the traditional high school, has no serious advocate at present. A system of "units" or "blocks" of work has very generally replaced it. This means that a "unit" of arithmetic is followed by a "unit" of geometry, or algebra, or trigonometry, according to the preference of the individual school.† In reality this is still "stratified" mathematics, but the strata have become thinner. This arrangement is called "general" (or "correlated", "composite", "unified") mathematics.

The effectiveness of this plan depends on many factors, especially on the amount of genuine correlation which it consciously realizes.‡ An actual "fusion" of the mathematical ingredients appears to be customary only in industrial and manual arts classes, usually in connection with practical problems or projects.‖

§Thorndike, E. L., and others. *"The Psychology of Algebra."* MacMillan Company, 1923.

¶See, for example, Dickinson and Ruch, *"An Analysis of Certain Difficulties in Factoring in Algebra,"* The Journal of Educational Psychology, May, 1925.

*Bennett, pp. 92-95.

†Breslich, E. *"The Unitary Organization of the Mathematics of the Seventh, Eighth, and Ninth Grades"*, Mathematics Teacher, April. 1923.

‡See *Report of the National Committee*, pp. 12-14; also, see Reeve, W. D. *"The Case for General Mathematics"*, Mathematics Teacher, November, 1922.

‖In this connection, it would be profitable for secondary teachers to study the corresponding development in the colleges. See, for example, Miss V. Sanford's article on *"Textbooks in Unified Mathematics for College Freshmen"*, Mathematics Teacher, April, 1923.

Of course, the creation of a single course in "general mathematics", suitable for all schools and all pupils, may prove to be as utopian as has been the development of a "general" method. What is needed, however, is a scientific theory of correlation, supplemented by flexible criteria of arrangement.

7. *Pedagogic Considerations.* For reasons of economy, a number of closely related items will be considered together under this heading.

a. *Methods of Teaching.* In general, better teaching is being done today in many junior high schools than in the average upper high school. The usual, mechanical "recitations" are disappearing. There is much more "motivation" than formerly. Drill work of a more intelligent and effective type is coming into use. The number of trained teachers is rapidly increasing.* At the same time, it is significant that only two monographs, of very moderate scope, have appeared thus far, which are devoted exclusively to mathematical methods in the junior high school.†

b. *Projects.* Much confusion has been caused by uncritical, indiscriminate eulogies of the "socialized recitation" and of "project teaching". A return to sanity is under way. A real "project" in mathematics takes very much time; it renders an organic development of a cumulative subject like mathematics almost impossible, within the customary meagre time allowance; it cannot be incorporated in a textbook; it requires expert teaching; and ordinarily it functions only in small classes, or in private schools. Nevertheless, supplementary projects, conducted by inspired teachers, are very desirable.

c. *Ability Grouping.* The National Committee offered the significant recommendation that "in the junior high school, comprising grades seven, eight, and nine, the course for these

*See Professor R. C. Archibald's scholarly report on the "*Training of Teachers of Mathematics*", Chap. XIV of the Report of the National Committee.

†Barber, Harry C. "*Teaching Junior High School Mathematics*". Houghton Mifflin Company, 1924; the second monograph, in pamphlet form, may be obtained by addressing its author, Dr. John R. Clark, of the Lincoln School of Teachers College.

three years should be planned as a unit *with the purpose of giving each pupil the most valuable mathematical training he is capable of receiving in those years, with little reference to courses which he may or may not take in succeeding years.*"

A one-sided interpretation of this statement might easily lead to a repetition of the deplorable blunder of the Committee of Ten (1893), when it refused to recognize the unavoidable distinction between "terminal" and "preparatory" courses. The psychology of individual differences, the use of survey tests, the introduction of a system of guidance in the larger schools, and the pre-vocational interests of many children, have conclusively disposed of the old conception of a single course for all. As a result, there have been numerous attempts at a more homogeneous classification of pupils, and differentiated courses are being developed.

It is too early to pass final judgment on the many related experiments which are being conducted throughout the country.‖ Perhaps it is safe to assert, however, that a *refined* system of ability grouping has proved impossible, for administrative reasons.‡ The logical consequence of an extreme form of classification would be individual instruction, after the fashion of the Dalton plan, which represents a curious reversion to the days of 'the little red school house".

d. *Use of Standardized Tests.* Much harm has been done by the premature, mechanical use of "standardized" tests.§ At best, the available tests can serve only a provisional or investigational purpose. The elaboration of really significant

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‖See Bruner, op. cit., Chap. III.

‡Equally futile is the attempt to enlarge the number of prescribed or elective courses in mathematics. For a statement of the thoroughly untenable ideas of Professor Snedden, see his article in the Mathematics Teacher, "*Prescribed versus Elective Mathematics in Junior High Schools*", February, 1922.

§See, especially, Upton, C. B. "*The Influence of Standardized Tests on the Curriculum in Arithmetic*", Teachers College Record, April, 1925; Walker, Helen M. "*What the Tests Do Not Test*", Mathematics Teacher, January, 1925; Professor Reeve's report on "*The Improvement of Tests in Mathematics*", in this volume; and the same author's latest work, "*A Diagnostic Study of the Teaching Problems in High-School Mathematics*", Ginn & Company, 1926.

tests will take much time, and it will presuppose the general acceptance of scientifically tested objectives, obviously not to be expected very soon. In the meantime, teachers should study the literature on objectives and on the newer types of tests. They should then begin to experiment independently in their own classrooms. The motto should be: "Objectives first, then teach, test, teach again, test again."

8. *The Arithmetic Situation.* An immense amount of work has been done in the field of reconstructed arithmetic.* Suggestions of reformers range all the way from the complete elimination of the subject in the junior high school to an exaggerated emphasis extending into the upper high school. Some feasible middle ground must be found. Reliable inventory tests have revealed a surprising weakness in the elementary processes and an extensive ignorance of the fundamental principles. Disregarding these weaknesses will not remedy them.† Perhaps the situation may be summarized by stating that the present practice of the best schools involves three typical features: 1) Carefully conducted, individualized drill work; 2) the essentials of socialized arithmetic of the community type; 3) the occasional use of supplementary projects.‡

A really scientific reorganization of arithmetic is of the utmost importance, since the necessary time for the newer elements of the curriculum can be provided only by the drastic elimination of all unessentials. In this direction, much remains to be acomplished. Some authors and syllabuses are still bent on making every pupil into an expert accountant, or a bookkeeper, or a banker. Others continue to show their preference for the encyclopaedic bulk of the older texts. It should be realized at last that a highly specialized content is

*See, for example, Monroe, Walter S. *"Principles of Method in Teaching Arithmetic"*, Chap. IV of the Eighteenth Yearbook of Nat. Soc. for the Study of Education, Part II.

†See Osburn, W. J. *"Corrective Arithmetic"*, Houghton Mifflin Company, 1924.

‡Lull and Wilson, *"The Redirection of High School Instruction"*, J. B. Lippincott Company, 1921, pp. 74-80.

out of place in a junior high school, and that the training in science and in the applied arts, which is offered to an increasing extent in good schools, renders a large part of the traditional arithmetic course both unnecessary and obsolete.

9. *The Case of Intuitive Geometry.* As was pointed out above, intuitive geometry has not yet received the attention to which it is entitled. Its precise function does not seem to be understood, and even the excellent suggestions given by the National Committee have apparently been insufficient. A further campaign of education may be necessary. Above all, would it not be possible, in connection with training courses for teachers, to present more clearly the compelling reasons for emphasizing this phase of mathematical training?

Historically, mathematics was called into existence because of the necessity for counting and measuring. This fact has given to mathematics a double foundation, namely arithmetic and geometry. It is apparent that we cannot make or manufacture the simplest article without giving due attention to its form, its dimensions, and the proper relation of its parts. Nature and the manual arts are readily seen to be the two permanent sources of geometry. More especially, training in space intuition and in plastic thinking is at the bottom of all forms of applied art. The geometric principles of equality, symmetry, congruence, and similarity, are implanted in the very nature of things. The art of measurement permeates the fabric of modern civilization at every point. Then, too, intuitive geometry has a functional aspect through the unique training which it affords in the discovery and formulation of relationships. Finally, intuitive geometry is absolutely essential as a preparation for effective work in demonstrative geometry.*

When thus conceived, intuitive geometry serves to vitalize and humanize the whole course in elementary mathematics. That these arguments have had a convincing force abroad, is proved by the constantly increasing attention which the leading European countries have given to elementary geometric instruction ever since the days of Pestalozzi, Herbart and Froebel.†

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*The writer has stated these arguments more completely in other publications. See "*Illustrated Mathematical Talks by Pupils of the Lincoln School*", pp. 23-32. New York, 1920.

†For a bibliography extending back to the year 1803, see the excellent monograph of the late Professor P. Treutlein, "*Der Geometrische Anschauungsunterricht*", Leipzig, 1911.

10. *The Reorganization of Algebra.* After much controversy, a new and greatly reduced algebra program is taking form. A majority of the recent writers would probably accept this formulation of the new content: "It should include the equation, the graph, and the formula, in a functional setting, together with as much purposeful technique as is needed for the solution of suitable problems."

Great uncertainty prevails, however, as to the significance and the scope of problem-solving.* Some writers seem perfectly willing to sacrifice the traditional verbal problems in favor of more intensive training in other abilities.† For obvious reasons, too, the "function concept" is still under constant debate. It is conceded that genuine work in functional thinking is certainly more appealing than much of the usual formalism of algebra, and less difficult. On the other hand, if functionality is to be an organizing principle of fundamental importance, the whole course must be readjusted accordingly.‡ Many authors still try to retain all the old material, and at the same time to glue on a few patches of functional work. It is this *double* burden, and *not* the function idea as such, which is causing many secondary teachers to view the new program as a sort of bête noire. In the opinion of the writer, a really organic algebra curriculum can be evolved only in a six-year high school.

11. *Unsettled Problems.* Needless to say, a complete enumeration of the many unsettled issues in the field of junior high school mathematics cannot be attempted in these pages. They involve the whole domain of aims, of curriculum reconstruction, of methods and standards.

Thus, much might be said of the stimulating influence which is to be expected from the inclusion of numerical trigonometry in the new curricula. Again, it would be an inviting task, if

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*See, especially, Ligda, Paul. "*The Teaching of Elementary Algebra*", Houghton Mifflin Company, 1925.

†See Thorndike, E. L. (and others), "*The Psychology of Algebra*", Chap. V.

‡See Hedrick, E. R. "*Functionality in Mathematical Instruction in Schools and Colleges*", Mathematics Teacher, April, 1922.

space permitted, to discuss the relation of demonstrative geometry to an ideal junior high school program.

Undoubtedly, many of the problems which are so perplexing at this time can be disposed of only by continued classroom experimentation. And, in the opinion of the writer, the six-year high school holds the key to the solution of not a few of them.

12. *The Message of Other Countries.* Mathematics, because of its cosmic background, has always transcended national and geographic barriers. Hence, the experience of other countries with reference to the organization of mathematical instruction cannot safely be ignored.

Teachers of mathematics have certainly been greatly benefited by constant reference to the reports of the International Commission. Since the return of peace, after the great cataclysm of 1914, many educational readjustments have occurred in practically all the leading countries of the world. The literature describing these changes is gradually becoming accessible.* From the best European schools we can learn the importance of better scholarship and greater thoroughness, while America sends back to the Old World the inspiration of broader horizons due to the vastly increased scope of its unique system of democratic education.

CONCLUSION

At the end of this brief survey, the writer is only too conscious of the very fragmentary character of his remarks. He regrets that many questions of importance had to be ignored entirely. Perhaps enough has been said, however, to convey the central message of the story, namely, that much progress has been made and that, on the whole, the picture is encouraging. Many unselfish men and women have assisted in making the junior high school what it is today. Its

*See, for example, Kandel, I. L. "*The Reform of Secondary Education in France*", pp. 118-126, Teachers College, New York, 1924; also Lietzmann, Dr. W. "*New Types of Schools in Germany and their Curricula in Mathematics*". (Transl. by V. Sanford). Mathematics Teacher, March, 1924.

many unsolved problems may confidently be left to the enterprise of a promising future.

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Part II

An extensive bibliography on the subject of junior high school Mathematics would exceed the limits of this Yearbook. For many valuable contributions the reader is referred to the following sources:

1. The Report of the National Committee.
2. The Mathematics Teacher.
3. School Science and Mathematics.
4. The School Review.
5. School and Society.
6. The Journal of Educational Psychology.
7. Teachers College Record.
8. The Journal of Educational Research.
9. The Yearbooks of the National Society for the Study of Education.
10. The Supplementary Educational Monographs of the University of Chicago.

SOME RECENT INVESTIGATIONS* IN ARITHMETIC

By Frank Clapp

During the last five years the writer has chosen to confine his research work largely to the field of arithmetic. Five major problems have been or are being given attention. Four of these problems are:

1. The Number Combinations: Their Relative Difficulty and the Frequency of their Appearance in Text Books.
2. A Plan for the Analysis and Scoring of Text Books in Arithmeti.
3. Elements of Difficulty in the Interpretation of Concrete Arithmetic.
4. The Knowledge which Pupils in the Different Grades have of the Fundamental Principles in Arithmetic.

The investigation of the first of these problems has been completed and the report is published as Bulletin No. 2. of the Bureau of Educational Research of the University of Wisconsin.

The study of the second problem is almost complete and will appear soon in form for use by superintendents and teachers.

The study of the third problem is complete and will appear sometime during the present year as Bulletin No. 8 of the Bureau of Educational Research of the University of Wisconsin.

The study of the fourth problem was begun only this year and will not be complete for another year.

This paper presents a summary of Problems 1, 2, and 3 as

*Detailed investigations and controlled experiments are distinctly the product of the last quarter of a century. The Yearbook would not be truly representative of the recent developments without a sampling of the newer types of materials that are developing to guide our practice.—*The Editors.*

stated above and sets forth briefly the procedure that is being used in the study of Problem 4.

In the study of the number combinations a total of 10,945 pupils were tested. The number of answers to combinations was 3,862,332. The first and second books of two series of arithmetics were analyzed.

Test A.—ADDITION

1	3	2	5	6	7	8	4	1	3
2	4	3	2	3	1	4	4	6	8
–	–	–	–	–	–	–	–	–	–
4	9	4	5	8	1	0	8	7	4
5	3	7	6	7	9	7	5	2	1
–	–	–	–	–	–	–	–	–	–
0	5	3	5	6	8	1	3	7	8
2	5	3	7	9	0	1	7	7	1
–	–	–	–	–	–	–	–	–	–
0	2	3	2	6	9	4	1	3	4
4	9	1	2	7	9	2	0	5	6
–	–	–	–	–	–	–	–	–	–
8	2	5	2	5	7	5	0	6	0
9	6	1	8	0	9	8	9	6	3
–	–	–	–	–	–	–	–	–	–
8	5	0	9	2	4	3	2	3	1
8	9	6	4	1	3	2	5	6	7
–	–	–	–	–	–	–	–	–	–
4	6	8	5	3	7	6	7	9	7
8	1	3	4	9	4	5	8	1	0
–	–	–	–	–	–	–	–	–	–
6	2	1	2	7	9	0	7	1	4
8	7	4	0	5	6	8	3	8	0
–	–	–	–	–	–	–	–	–	–
9	1	7	2	0	5	6	9	6	1
2	3	6	4	1	3	4	8	2	5
–	–	–	–	–	–	–	–	–	–
8	0	9	8	9	3	9	6	4	0
2	5	7	5	0	0	5	0	9	0
–	–	–	–	–	–	–	–	–	–

In determining the relative difficulty of the number combinations pupils were tested in two ways: First, the combinations were presented to the pupils in single form as is shown in the preceding table.

The combinations listed above were read to pupils at the approximate rate of one combination each two seconds. The rate was determined by experimentation and the time was made short enough to prevent a pupil's counting or getting the answer in any other round-about way. Pupils wrote their answers on specially prepared sheets. The purpose of the study was to determine which combinations had been reduced to the automatic level. The results may be said to indicate

TEST B—ADDITION—PUPIL'S SHEET

Combinations in Problems

(One)	(Two)	(Three)	(Four)
1	6	5	4
3	3	4	9
5	6	0	4
4	2	4	1
0	4	4	7
6	7	5	3
7	5	2	4
3	5	8	6
-	—	—	—
(Five)	(Six)	(Seven)	(Eight)
0	1	7	7
2	5	6	2
3	8	5	3
0	9	5	8
7	5	9	8
5	9	9	0
3	6	1	4
7	4	9	6
-	—	—	—
(Nine)	(Ten)	(Eleven)	(Twelve)
2	78	60	38
3	76	81	53
8	99	56	03
7	37	63	91
0	86	97	49
8	54	97	98
3	39	35	51
7	61	65	49
-	—	—	—
	(Thirteen)	(Fourteen)	
	22	0	
	14	12	
	01	85	
	67	12	
	24	82	
	18	68	
	24	71	
	76	29	
	—	—	

the relative learning difficulty of the combinations.

Second, the combinations were presented to pupils in examples as shown on this page.

In this test it will be noticed that the sum of the first two numbers where the pupil starts to add (at the bottom) is 10. From this point the pupil meets, e.g. 5 + 8, as 15 + 8, or 25 +8, or 35+8, etc. In other words the basic addition combinations appear as secondary combinations.

Pupils were tested individually, the pupil adding aloud and giving each step while the teacher checked the results on sheets especially prepared for that purpose. If the pupil gave any step wrong he was set right at once and allowed to proceed. In this way the teacher could determine which combinations were missed. There was no time limit, the pupil taking all the time he needed for each combination and getting the answer in any way that he might choose.

The same general procedure as is indicated above for addition was used for subtraction, multiplication, and division in both Test A and Test B.

Bulletin No. 2 presents the results of both tests by grades,—Grades 3 to 8 in addition and subtraction, and 4 to 8 in multiplication and division for Test A, and Grades 4 to 8 in all processes for Test B. The Bulletin also gives the results for these grades combined.

One very important outcome of the investigation is the fact that the relative difficulty of the combinations presented in the two ways is not the same. Table I below, gives the one hundred addition combinations arranged in decreasing order of difficulty according to the two tests. The combinations are arranged in four groups, the hardest twenty-five being indicated by (1), the next hardest by (2), etc.

Note: The author is naturally primarily interested in the investigational aspects, but there are important pedogogical implications. Teachers have used this list for diagonisis to discover what combinations have not been mastered. For example, an eighth grade boy made a very low mark on standardized multiplication tests. His teacher asked him to give the products for the exercises in columns A and B under multiplication in the table on the following pages. It was discovered that this boy lacked only seven multiplication combinations. The realization that he had only seven facts to learn instead of a hopelessly large number was a new challenge. In a few days he was able to make an excellent showing on three different multiplication tests.—The Editors.

TABLE I

A list of the combinations in decreasing order of difficulty when presented singly (in the A columns) and when presented in examples (in the B columns).

Addition		Subtraction		Multiplication		Division	
A	B	A	B	A	B	A	B
(1)	(1)	(1)	(1)	(1)	(1)	(1)	(1)
8+5	4+9	14−9	11−3	7×0	0×2	2÷2	49÷7
7+9	9+6	13−4	16−9	0×5	9×4	9÷9	54÷6
5+8	6+7	16−9	8−4	0×7	9×7	28÷4	45÷9
9+7	8+6	14−5	11−4	0×1	8×6	1÷1	27÷3
6+8	6+9	17−9	14−9	4×0	8×8	36÷4	56÷7
6+9	3+9	15−9	7−0	0×8	7×6	8÷8	28÷4
5+7	5+8	17−8	9−1	0×2	0×5	3÷3	24÷4
7+8	7+9	15−7	15−9	0×4	0×7	6÷6	36÷6
8+7	5+9	13−9	9−0	2×0	6×9	56÷8	48÷6
9+6	8+5	13−5	11−2	0×6	7×8	54÷6	48÷8
5+9	9+9	16−7	11−7	9×0	4×8	7÷7	21÷7
8+9	7+5	13−7	15−8	8×0	0×8	5÷5	18÷2
8+6	4+8	13−8	13−5	0×3	7×4	4÷4	21÷3
4+7	2+9	11−3	16−7	6×0	9×0	63÷9	36÷4
7+5	7+6	14−6	9−9	0×9	8×7	0÷5	72÷9
4+9	3+8	12−3	14−5	3×0	7×9	0÷9	63÷9
9+5	7+7	15−6	13−8	9×7	8×4	54÷9	40÷8
9+4	6+6	14−8	10−2	5×0	6×2	0÷2	32÷4
6+7	4+7	12−5	13−9	1×0	0×9	0÷7	35÷7
5+6	6+8	12−7	13−4	8×7	9×8	0÷8	35÷5
4+6	9+7	11−4	5−0	9×6	0×3	0÷3	18÷3
7+6	8+8	11−6	5−4	6×9	6×8	64÷8	15÷3
7+4	9+3	16−8	6−0	7×9	0×1	48÷8	45÷5
9+8	9+8	18−9	9−3	7×8	8×9		
3+7	8+9	11−9	17−9	6×8	8×3		
(2)	(2)	(2)	(2)	(2)	(2)	(2)	(2)
9+0	1+3	11−7	6−3	8×6	8×0	0÷4	42÷7
2+6	8+7	13−6	6−6	4×9	7×3	0÷6	54÷9
9+3	5+7	15−8	13−6	4×7	7×5	42÷7	30÷6
0+6	7+8	8−5	2−2	1×1	6×7	36÷9	28÷7
6+5	2+2	11−5	8−8	8×9	9×6	28÷7	24÷3
3+8	3+6	11−8	11−5	9×4	6×6	63÷7	27÷9
3+4	8+3	9−2	6−4	7×4	5×1	27÷3	64÷8
3+9	7+4	10−3	10−7	7×6	6×4	24÷8	81÷9
2+3	9+4	10−9	12−5	8×8	7×7	48÷6	36÷9
3+5	1+9	14−7	8−1	8×5	7×1	24÷4	18÷6

TABLE I (Continued)

ADDITION		SUBTRACTION		MULTIPLICATION		DIVISION	
A	B	A	B	A	B	A	B
6+3	4+4	12−4	14−6	8×4	4×0	2÷1	63÷7
7+3	1+8	10−8	5−1	9×8	0×6	49÷7	42÷6
2+7	2+8	7−4	11−8	9×3	9×9	40÷8	56÷8
8+4	3+7	12−8	14−7	5×6	9×3	72÷8	16÷2
4+8	2+6	1−0	7−1	6×7	3×0	18÷6	24÷8
8+0	3+5	8−3	9−6	4×8	4×9	24÷3	72÷8
1+0	9+5	7−0	4−1	3×8	7×0	72÷9	16÷4
5+2	1+4	12−9	15−6	4×6	9×5	9÷1	20÷4
4+2	1+5	10−5	7−5	5×8	8×5	24÷6	30÷5
1+2	1+7	10−6	2−1	3×9	0×4	0÷1	14÷7
5+3	9+2	8−0	7−3	7×7	9×2	6÷3	14÷2
0+3	4+5	2−0	8−2	3×7	3×2	32÷4	25÷5
0+5	3+4	9−7	15−7	5×7	5×5		
5+1	6+2	9−4	7−2	5×9	5×6		
7+0	1+6	4−0	12−4	9×5	5×8		
(3)	(3)	(3)	(3)	(3)	(3)	(3)	(3)
8+7	4+6	3−0	10−1	7×3	4×6	18÷2	12÷3
0+1	2+7	5−0	9−2	9×9	4×4	8÷4	12÷6
7+2	6+3	9−6	9−8	7×5	4×7	56÷7	32÷8
1+9	6+5	8−2	9−4	4×5	3×1	45÷9	8÷4
0+5	1+2	10−4	10−3	3×4	6×1	30÷6	9÷3
8+1	5+3	8−8	13−7	2×4	3×6	3÷1	12÷2
6+2	2+5	9−3	12−7	4×2	2×0	35÷7	16÷8
0+4	2+4	7−3	8−7	5×3	6×3	18÷3	10÷5
3+6	3+3	6−0	7−6	6×6	5×9	8÷1	4÷2
0+2	5+6	9−0	6−1	6×3	2×8	45÷5	12÷6
2+4	7+3	7−2	3−0	6×4	9×1	42÷6	5÷1
3+0	5+5	12−6	2−0	1×2	5×7	18÷9	15÷5
4+5	4+3	10−7	14−8	6×5	3×9	16÷8	18÷9
0+8	6+0	8−6	12−8	2×6	4×3	4÷1	6÷6
6+0	8+4	11−2	12−3	3×6	3×7	20÷4	20÷5
8+3	5+4	10−1	6−2	8×3	2×4	6÷1	2÷2
8+2	1+1	6−2	8−3	6×2	6×5	32÷8	10÷2
6+4	8+1	10−2	12−9	3×5	5×3	35÷5	6÷3
1+4	6+4	7−6	16−8	8×1	5×0	27÷9	40÷5
9+1	2+3	9−5	9−7	1×9	1×1	5÷1	8÷8
5+0	0+9	8−7	9−5	4×3	4×1	16÷4	7÷7
6+6	0+6	4−1	8−6	5×5	5×4	7÷1	12÷4
3+2	7+2	2−2	6−5	3×3	2×3	12÷4	6÷2
4+3	0+1	7−5	1−0	9×1	2×9		
1+5	3+1	6−4	11−9	7×2	4×2		

Table I (Continued)

Addition		Subtraction		Multiplication		Division	
A	B	A	B	A	B	A	B
(4)	(4)	(4)	(4)	(4)	(4)	(4)	(4)
7+1	3+2	6+1	17−8	2×7	2×1	40÷5	9÷1
2+9	1+0	2−1	8−5	2×3	2×2	21÷7	4÷4
2+5	6+1	5−2	18−9	4×1	6×0	21÷3	0÷3
2+8	7+1	6−3	7−7	4×4	3×3	30÷5	3÷3
4+4	0+7	3−1	3−1	5×2	8×2	12÷3	5÷5
4+1	0+4	7−1	5−2	7×1	8×1	15÷5	0÷4
3+1	0+3	4−3	1−1	8×2	1×0	12÷6	9÷9
4+0	4+2	5−5	5−3	3×2	3×4	36÷6	0÷7
1+8	2+0	8−4	10−4	3×1	1×2	16÷2	0÷6
9+9	3+0	6−5	7−4	6×1	3×8	14÷7	0÷5
7+7	0+5	9−8	4−0	1×8	4×5	20÷5	6÷1
2+0	4+0	4−4	4−2	5×4	1×9	8÷2	0÷8
6+1	0+8	9−1	8−0	2×9	1×8	12÷2	1÷1
5+4	8+2	9−9	5−5	2×1	2×6	15÷3	0÷9
3+3	5+2	5−3	11−6	1×6	1×3	14÷2	8÷1
1+1	5+0	5−4	10−5	1×5	7×2	10÷5	2÷1
9+2	4+1	1−1	3−3	2×2	1×5	6÷2	0÷1
8+8	7+0	8−1	4−3	9×2	2×5	81÷9	8÷2
1+3	2+1	4−2	4−4	2×8	3×5	9÷3	0÷2
1+6	9+1	6−6	10−6	1×4	2×7	10÷2	4÷1
1+7	5+1	3−3	3−2	2×5	1×8	25÷5	7÷1
2+1	0+2	3−2	10−9	1×3	1×7	4÷2	3÷1
2+2	9+0	7−7	10−8	5×1	1×4		
5+5	8+0	5−1	12−6	1×7	5×2		
0+0	0+0	0−0	0−0	0×0	0×0		

In the first (hardest) group of twenty-five combinations in Table I, Column B, are found the following: —6 + 6, 7 + 7, 8 + 8, 9 + 9. In Column A, however, the last three of these are found in the fourth (easiest) group, and the first one is in the fourth place from the end of the third group. In multiplication all of the zero combinations except 0 × 0 occur before any of the other combinations in Column A, but this is not the case in Column B. In division, eleven combinations in the first group in Column A occur in the fourth group in Column B. (Note especially 2 ÷ 2 and other similar combinations.) Many other comparisons may be made all of which point to the fact that the difficulties which pupils en-

counter with the combinations are not the same when they are presented singly and when they are presented in examples.

The Pearson *Coefficients* of Correlation between the difficulty of combinations presented in the two ways for the different processes are as follows:

Addition	.544
Subtraction	.390
Multiplication	.580
Division	.022

These coefficients correspond roughly to the relative degrees to which the pupils have mastered the combinations in the four processes.

In an editorial in the Journal of Educational Research for December, '25, Buckingham presents coefficients of correlation between the difficulty of combinations as found in this study and as found in the study by Holloway* and also by Smith.† Holloway confined his attention to the forty-five combinations in addition and multiplication regarding, for example, $2 + 5$ as the same as $5 + 2$ and 2×5 as the same as 5×2. Buckingham's coefficients between Holloway's results and the results of the present study are .670 for addition and .840 for multiplication. Smith used a very small number of pupils but used the one hundred combinations in addition, subtraction, and multiplication and the ninety in division as was done in the study at Wisconsin. Buckingham's coefficients between Smith's results and the writer's are as follows:

Addition	.780
Subtraction	.710
Multiplication	.790
Division	.790

Unfortunately Buckingham does not state whether in determining his correlation he used the results of the A or of

*Holloway, H. V. An Experimental Study to Determine the Relative Difficulty of the Elementary Number Combinations in Addition and Multiplication. Ph. D. Thesis, University of Pennsylvania, June, 1914.

†Smith, J. H. Arithmetical Combinations, *Elem. Sch. Jl. 21*, pp. 762-70.

the B tests from the present study. Neither does he state whether in the case of the Holloway study he used from the writer's study the same grades used by Holloway or not.

Teachers and text books have for a long time spoken of forty-five number combinations. In doing so the zero combinations are neglected and it is assumed, e. g., that 3+9 is the same and 9+3, and 3×9 the same as 9×3.

Preceding tables have made it clear that the zero combinations are not a negligible consideration when accuracy is desired. The question as to the identity of the two numbers of a "pair" of combinations, e. g., 3+9 and 9+3 may be discussed on the basis of their identity as a stimulus in connection with a number habit or on the basis of their equality of difficulty when the answers are determined by some thinking procedure. In either case it seems that if the two combinations are identical, they should have the same percentage of error and also the individual pupils who miss the one should miss the other. That such is not the case appears in the following tables. Table II gives the percentage of error for each of the two members of all the pairs of combinations in addition (on the left) and then (on the right) gives one of the two members paired with a combination to which it is more nearly equal in difficulty than it is to its mate. For example, it will be seen in the first line of the table that 0 + 1 has a percentage of error of 4.7 while 5.3 is the percentage for 1 + 0, and that 0 + 1 and 0 + 7 have exactly the same percentage of error. The table is for the A tests.

In only two cases, 4 + 8 and 8 + 4, and 5 + 8 and 8 + 5 are the percentages of difficulty more nearly equal for the two members of a pair of combinations than for one of the members and some other combination. Not only is this true but the percentages for the two members of pairs often differ very materially.

The A tests, it will be recalled, measure the degree of automatization for the combinations. Since this is the case, it appears from Table II that the two members of a "pair" of combinations are not identical as stimuli in connection with

TABLE II

SHOWING THAT MEMBERS OF "PAIRS" OF COMBINATIONS (e. g., 3+5 and 5+3) ARE NO MORE ALIKE IN DIFFICULTY THAN ARE OTHER COMBINATIONS. BASED ON THE A TESTS

Pairs of Combinations				Combinations of Near Equal Difficulty			
Combination	Per cent of error	Per cent of error	Combination	Combination	Per cent of error	Per cent of error	Combination
0+1	4.7	5.3	1+0	0+1	4.7	4.7	0+7
0+2	4.4	2.6	2+0	0+2	4.4	4.4	3+6
0+3	5.0	4.3	3+0	0+3	5.0	4.9	0+9
0+4	4.4	2.8	4+0	0+4	4.4	4.4	6+2
0+5	4.6	3.6	5+0	0+5	4.6	4.6	7+2
0+6	6.5	3.9	6+0	0+6	6.5	6.5	6+5
0+7	4.7	4.8	7+0	0+7	4.7	4.5	8+1
0+8	4.0	5.4	8+0	0+8	4.0	3.9	8+3
0+9	4.9	7.5	9+0	0+9	4.9	4.9	5+1
1+2	5.2	1.8	2+1	1+2	5.2	5.2	5+3
1+3	2.1	2.9	3+1	1+3	2.1	2.1	1+6
1+4	3.7	3.0	4+1	1+4	3.7	3.7	9+1
1+5	3.4	4.9	5+1	1+5	3.4	3.6	5+0
1+6	2.1	2.5	6+1	1+6	2.1	1.8	1+3
1+7	1.8	3.3	7+1	1+7	1.8	1.8	2+1
1+8	2.7	4.5	8+1	1+8	2.7	2.8	4+0
1+9	4.6	3.7	9+1	1+9	4.6	4.7	0+1
2+3	6.3	3.5	3+2	2+3	6.3	6.2	3+5
2+4	4.4	5.2	4+2	2+4	4.4	4.4	0+2
2+5	3.1	5.3	5+2	2+5	4.1	3.0	4+4
2+6	7.3	4.4	6+2	2+6	7.3	7.0	9+3
2+7	5.8	4.6	7+2	2+7	5.8	5.8	8+4
2+8	3.1	3.9	8+2	2+8	3.1	3.3	7+1
2+9	3.3	2.4	9+2	2+9	3.3	3.0	4+1
3+4	6.4	3.4	4+3	3+4	6.4	6.4	3+9
3+5	6.2	5.2	5+3	3+5	6.2	6.0	6+3
3+6	4.4	6.0	6+3	3+6	4.4	4.4	2+4
3+7	7.6	5.9	7+3	3+7	7.6	7.7	9+8
3+8	6.5	3.9	8+3	3+8	6.5	6.5	0+6
3+9	6.4	7.0	9+3	3+9	6.4	6.3	2+3
4+5	4.2	2.5	5+4	4+5	4.2	4.3	3+0
4+6	8.6	3.8	6+4	4+6	8.6	8.7	5+6
4+7	10.0	8.0	7+4	4+7	10.0	9.5	7+5
4+8	5.8	5.8	8+4	4+8	5.8	5.8	8+4
4+9	9.1	9.0	9+4	4+9	9.1	9.0	9+5
5+6	8.7	6.5	6+5	5+6	8.7	8.6	4+6
5+7	11.8	9.5	7+5	5+7	11.8	11.7	7+8
5+8	14.9	15.0	8+5	5+8	14.9	15.0	8+5
5+9	10.9	9.0	9+5	5+9	10.9	10.8	8+9
6+7	9.0	8.1	7+6	6+7	9.0	9.0	9+4
6+8	13.8	10.8	8+6	6+8	13.8	14.2	9+7
6+9	13.1	11.5	9+6	6+9	11.5	11.6	8+7
7+8	11.7	11.6	8+7	7+8	11.7	11.8	5+7
7+9	15.0	14.2	9+7	7+9	15.0	15.0	8+5
8+9	10.8	7.7	9+8	8+9	10.8	10.8	8+6

number habits.

It might be that the pupils who missed the first member of a pair of combinations were not among those who missed the

second member. If this were the case, it might be concluded that missing such combinations was largely or wholly a matter of chance. It is of significance then to know: (1) To what extent pupils missed both members of pairs; (2) To what extent they missed the first member and not the second; and (3) To what extent they missed the second and not the first.

TABLE III

THE PERCENTAGE OF PUPILS MISSING: (1) BOTH MEMBERS OF PAIRS OF COMBINATIONS; (2) THE FIRST BUT NOT THE SECOND; (3) THE SECOND BUT NOT THE FIRST

Pairs of Combinations		Missed both	Missed 1st not 2nd	Missed 2nd not 1st	Pairs of Combinations		Missed both	Missed 1st not 2nd	Missed 2nd not 1st
1st	2nd				1st	2nd			
2+1	1+2	.36	.54	1.98	7+3	3+7	1.74	2.46	3.40
4+3	3+4	1.02	.72	2.94	1+8	8+1	.48	.60	2.10
3+2	2+3	.60	.78	2.94	4+0	0+4	.36	.48	2.22
2+5	5+2	1.08	1.08	2.46	9+2	2+9	.42	.90	1.32
3+6	6+3	1.02	1.20	3.06	1+3	3+1	.42	.78	1.20
1+7	7+1	.30	.90	1.38	7+6	6+7	2.40	3.72	3.90
4+8	8+4	1.32	3.48	2.76	2+4	4+2	.84	1.80	2.16
6+1	1+6	.24	1.14	.72	0+1	1+0	1.08	1.56	2.04
8+3	3+8	1.14	1.80	4.08	5+3	3+5	1.38	1.86	2.88
5+4	4+5	.54	1.08	1.56	6+4	4+6	1.02	.78	5.28
3+9	9+3	1.38	3.72	3.96	9+8	8+9	2.94	2.34	5.52
7+4	4+7	2.70	3.18	5.40	6+2	2+6	.96	1.62	3.48
6+5	5+6	1.68	2.22	4.14	1+5	5+1	.54	.96	1.62
7+8	8+7	5.40	4.26	6.42	8+2	2+8	.66	1.02	1.38
9+1	1+9	.48	.96	1.98	0+5	5+0	.48	1.74	1.44
7+0	0+7	1.32	1.50	2.22	9+7	7+9	6.06	4.62	6.96
6+8	8+6	5.46	7.38	3.78	8+5	5+8	5.52	6.02	6.60
2+7	7+2	.72	3.24	1.86	9+0	0+9	.90	2.94	1.80
1+4	4+1	.30	1.50	1.08	3+0	0+3	.66	1.38	2.52
2+0	0+2	.30	.72	1.76	9+5	5+9	2.70	2.88	6.38
7+5	5+7	2.46	4.74	5.76	6+0	0+6	.48	1.56	2.88
9+6	6+9	4.92	3.66	6.06	4+9	9+4	2.64	4.14	3.86
0+8	8+0	1.08	1.56	2.04	.....	.....	.....	.....	.....

These questions are answered in Table III for 1750 pupils in connection with the A tests for addition, the pupils being in Grades 3 to 8 inclusive.

TABLE IV

THE PERCENTAGE OF PUPILS MISSING: (1) BOTH MEMBERS OF PAIRS OF COMBINATIONS; (2) THE FIRST BUT NOT THE SECOND; (3) THE SECOND BUT NOT THE FIRST

Pairs of Combinations		Missed both	Missed 1st not 2nd	Missed 2nd not 1st	Pairs of Combinations		Missed both	Missed 1st not 2nd	Missed 2nd not 1st
1st	2nd				1st	2nd			
2+1	1+2	.36	.54	1.98	7+3	3+7	1.74	2.46	3.40
4+3	3+4	1.02	.72	2.94	1+8	8+1	.48	.60	2.10
3+2	2+3	.60	.78	2.94	4+0	0+4	.36	.48	2.22
2+5	5+2	1.08	1.08	2.46	9+2	2+9	.42	.90	1.32
3+6	6+3	1.02	1.20	3.06	1+3	3+1	.42	.78	1.20
1+7	7+1	.30	.90	1.38	7+6	6+7	2.40	3.72	3.90
4+8	8+4	1.32	3.48	2.76	2+4	4+2	.84	1.80	2.16
6+1	1+6	.24	1.14	.72	0+1	1+0	1.08	1.56	2.04
8+3	3+8	1.14	1.80	4.08	5+3	3+5	1.38	1.86	2.88
5+4	4+5	.54	1.08	1.56	6+4	4+6	1.02	.78	5.28
3+9	9+3	1.38	3.72	3.96	9+8	8+9	2.94	2.34	5.52
7+4	4+7	2.70	3.18	5.40	6+2	2+6	.96	1.62	3.48
6+5	5+6	1.68	2.22	4.14	1+5	5+1	.54	.96	1.62
7+8	8+7	5.40	4.26	6.42	8+2	2+8	.66	1.02	1.38
9+1	1+9	.48	.96	1.98	0+5	5+0	.48	1.74	1.44
7+0	0+7	1.32	1.50	2.22	9+7	7+9	6.06	4.62	6.96
6+8	8+6	5.46	7.38	3.78	8+5	5+8	5.52	6.02	6.60
2+7	7+2	.72	3.24	1.86	9+0	0+9	.90	2.94	1.80
1+4	4+1	.30	1.50	1.08	3+0	0+3	.66	1.38	2.52
2+0	0+2	.30	.72	1.76	9+5	5+9	2.70	2.88	6.38
7+5	5+7	2.46	4.74	5.76	6+0	0+6	.48	1.56	2.88
9+6	6+9	4.92	3.66	6.06	4+9	9+4	2.64	4.14	3.86
0+8	8+0	1.08	1.56	2.04	.....	.....	.....	.....	.....

The majority of the pupils got both members of the pairs right. Now, if a larger percentage miss both than miss one or the other, the indentity of the two would be fairly well estab-

lished. This, however, is not the case. In every one of the forty-five cases in Table III the percentage for one of the members is larger than that for both. It seems, then, that the two members are not identical since they are not at all equal in difficulty.

It remains to be seen whether the same conclusions may be drawn from the results of the B tests. This appears in Table IV which is for addition, the percentages being based on the work of pupils in Grades 4 to 8 inclusive. The table is of the same character as Table II.

No discussion of Table IV seems necessary except to point out that in only one case, that of $8 + 9$ and $9 + 8$ are the percentages more nearly equal for the two members of a pair of combinations than for one of the members and some other combination. The table indicates that the members of pairs of combinations are no more nearly equal in difficulty than are other combinations when pupils are free to work out the answers in any way they like, just as was the case when automotization was tested.

ANALYSIS OF TEXTBOOKS

In the analysis of text books the question for which an answer was sought was, "If a pupil should do all of the drill work and solve all of the concrete problems in the text how many times would he think the answer to each of the combinations?" In order to be able to answer this question each example and each problem was solved and the combinations met in each one were listed.

The coefficients of correlation between the difficulty of the combinations as determined by the B test (combinations presented in examples, with no time limit) and the frequency of their appearance in the two text books were as follows:

	A	B
Addition	—.452	—.532
Subtraction	—.329	—.277
Multiplication	—.384	—.465
Division	—.421	—.538

These coefficients are all negative and of material size. It seems that the typical text book is not to be depended upon to provide the most effective material for the mastery of the number combinations.

In connection with Problem 2 the writer had the assistance of Mr. Ivan Swancutt, principal of the City High School, Wauwautosa, Wisconsin. The purpose in the preparation of this material was to devise a means whereby the real character of arithmetic text books might be determined and expressed in comparable terms. A score card for judging the merits of a text book was developed. The score on a general item is the sum of the scores for the detailed items, which make up the general item. A perfect score is 1000.

The analysis of the books as called for by the plan is rather detailed but a committee of five teachers is able to analyze and score one book in about two hours. Nine series of texts have been analyzed. A comparison of the analyses is almost startling. The score card and the figures giving these comparisons cannot be presented here because of limitation of space.

The Difficulty of Verbal Problems In Arithmetic

Problem 3 was investigated by Dr. L. L. Hydle, now at the University of Colorado, under the writer's supervision. The general problem as stated above is "Elements of Difficulty in the Interpretation of Concrete Problems in Arithmetic." The following eight elements were studied:

1. The visualization of the objects mentioned in the problems and of the general objective setting of the problem.
2. The size of numbers presented in the problem,—without reference to the accuracy or computation.
3. Unfamiliar terms, the understanding of which was not essential to the interpretation of the problem.
4. Mind set or sequence of like problems.
5. Non-essential numerical terms.
6. Names of objects familiar to pupils but difficult to visual-

ize as compared with names of objects less familiar to pupils but more difficult to visualize.

7. Problems presented in the form of projects as compared with problems presented in the formal arithmetical manner.
8. The use of symbols instead of the names of concrete objects.

It will be noticed that the eight theses have to do in large part with visualization. In fact the general thesis might be stated as follows,—"How visualization affects the interpretation of concrete problems in airthmetic."

Hydle's investigation involved a total of 8000 pupils enrolled in 46 different school systems in ten different states. His general method was to prepare five problems, each of which embodied the element that was supposed to be difficult and then to prepare five other problems, each of which was exactly like one in the first list except that it contained the element of difficulty in a less degree. The pupils in each school were divided into two groups of equal ability in arithmetic on the basis of tests. One set of five problems was given to one of these groups and the companion set to the other. Hydle's tables give the per cent of error made by each group on each pair of problems. He prepared a different set of problems for Grades 4 and 5 and for Grades 6, 7, and 8.

The following set of problems for Grades 6, 7, and 8 were used in the study of the first thesis, that is "the visualization of the objects mentioned in the problems and of the general objective setting of the problem."

SET A

Grades 6, 7, and 8

1. A merchant bought 80 rubber balls for his Christmas trade. He put them in 4 drawers, placing the same number in each drawer. He sold all the balls in 2 of the drawers and gave 2 of the remaining balls to his own children. How many balls had he left?
2. The American Book Company shipped 450 books to Supt. Smith in 18 boxes, putting the same number in each box; 3 boxes were lost and 10 books were stolen out of another box. How many books did Supt. Smith receive?

3. The cafeteria uses 50 grape fruits each week day and 75 grape fruits on Sunday. How many grape fruits will it use in 6 weeks?
4. Mr. Brown had 480 sheep on his ranch. He decided to ship some of them to Chicago. He secured 11 empty cars and shipped 30 sheep in each car. Then he bought 120 sheep from Mr. Jackson. How many sheep had he then?
5. A man had $500 to invest in the tailor business. He decided to make 7 overcoats. He found that the cloth in each overcoat cost $10, labor $5, lining and buttons $3. After paying the cost of 7 overcoats, how much money had he left?

SET B

Grades 6, 7, and 8

1. A dealer bought 80 tons of hay. He placed an equal amount of hay in each of his 4 storerooms. At the end of the month he had sold all the hay in 2 of the storerooms and had used 2 tons for the feeding of his own horses. How many tons of hay had he left?
2. Mr. Johnson shipped 450 quarts of cream in 18 cream cans, putting the same amount of cream in each can; 3 cans were lost and 10 quarts were stolen from another car. How many quarts of cream arrived safely?
3. A cafeteria uses 50 pounds of meat each week day and 75 pounds on Sunday. How many pounds of meat will it use in 6 weeks.
4. Mr. Brown started to drive from his home to Madison, Wisconsin, a distance of 480 miles. The first day he drove 11 hours at a rate of 30 miles per hour. That evening he met his friend, Mr. Jackson, whose home was located 120 miles on the other side of Madison. The next day he drove to Mr. Jackson's home. How many miles must he drive the next day?
5. A student had $500 in the bank. He paid $10 per week for board, $5 per week for roo mand $3 for laundry and amusements. How much money had he left after 7 weeks?

In the above problems the objects that are supposed to differ in the ease with which they may be visualized are as follows, Set A including objects that are relatively easy to visualize while those in Set B are difficult to visualize.

	Set A	*Set B*
1.	"rubber balls"	"tons of hay"
2.	"books" and "boxes"	"quarts of cream" in bulk
3.	"grape fruit"	"pounds of meat"
4.	"sheep" and "cars"	"miles" and "hour"
5.	"cloth, labor, lining and buttons" combined to make an "overcoat"	"board, room, laundry, amusements" combined to represent a total expenditure for one "week"

The results for these five problems are given in Table V below.

TABLE V.

Summary of Results for First Test or Form 4 Grades 6, 7, and 8

	Grade 6		Grade 7		Grade 8		Grades 6, 7, and 8	
	Number of Pupils		Number of Pupils		Number of Pupils		Number of Pupils	
	Group A.	Group B.	Group A.	Group B.	Group A.	Group B.	Group A.	Group B.
	701	689	789	872	650	636	2140	2197
	Per cent Wrong		Per cent Wrong		Per cent Wrong		Per cent Wrong	
Problem	Group A.	Group B.	Group A.	Group B.	Group A.	Group B.	Group A.	Group B.
1.	49.8	59.2	32.6	46.0	16.6	24.7	33.4	44.4
2.	65.5	78.8	50.8	62.6	35.5	45.3	50.9	62.7
3.	73.6	72.0	52.0	60.1	39.4	41.5	55.2	58.4
4.	47.8	83.8	32.2	71.9	18.0	60.5	33.0	72.3
5.	54.2	60.4	35.2	43.8	21.2	22.8	37.2	42.5
All Problems	58.2	70.8	40.6	56.9	26.1	39.0	41.9	56.1

It will be noticed in the above table that there is only one exception to the general trend of the results. This is in the case of Problem 3 for Grade 6. Hydle explains this discrepancy with the suggestion that the younger pupils in the smaller classes were perhaps unfamiliar with grape fruit. The differences in the percentages for the pairs of problems as given in the table are many times larger than the probable error of the differences according to Hydle.

It is imposible to present more of Hydle's results in this connection, but as stated above the study will appear in the near future as a Bulletin of the Bureau of Educational Research of the University of Wisconsin.

This study by Hydle, which constitutes his Doctor's thesis, is, in the judgment of the writer, a real contribution to the psychology of arithmetic. His findings in connection with practically all of his eight theses are consistent in the different grades and for all of the problems and in most cases the differences in the percentages are material.

The fifth problem, "The Knowledge which Pupils in the Different Grades have of the Fundamental Principles in Arithmetic", is an attempt to ascertain, not what kind or how difficult problems pupils can solve, but rather to what extent they appear to understand the basic character of the different phases of the subject.

The study will be state wide and is to be carried on through the Elementary Principals' Section of the State Teachers Association. It was begun during January of the present year.

This problem may be illustrated by reference to a study conducted by Superintendent H. W. Kircher of the Public Schools, Sheboygan, Wis. Superintendent Kircher gave the following test to 132 pupils who were finishing the work of the eighth grade.

TEST

1. One cent is what per cent of one dollar? .....
2. Find one-half per cent of two dollars. .....
3. Change 1/2% to a common fraction. .....
4. Five cents is five per cent of what? .....
5. Change 7/5 to per cent. .....
6. One cent is what per cent of two dollars? .....
7. 75% of a number is 6, what is the number? .....
8. 110% of a number is 220, what is the number? .....
9. 5 1/2% of a number is 44, what is the number? .....
10. A man lost 15% of his money and saved the remainder. What per cent of his money did he save? .....

The following table gives the percentage of pupils missing each of the problems.

Problem	*Pct. of Pupils Missing*	*Problem*	*Pct. of Pupils Missing*
1.	63	6.	77
2.	85	7.	54
3.	86	8.	71
4.	56	9.	80
5.	71	10.	26

The character of the wrong answers is indicated in Table VI below. The lack of space prevents the publication of more than sample results. For this purpose the responses on exercises 1, 2 and 3 have been chosen.

TABLE VI.

KINDS OF WRONG ANSWERS FOR EACH PROBLEM WITH NUMBER OF TIMES ANSWER OCCURRED
(N. A. means no answer)

Problems:

1		2		3	
1/100%	:34	$1.00	:60	½	:49
.01	21	100%	22	1/5	14
100%	10	50%	6	n. a.	12
99%	6	n. a.	6	.005	9
n. a.	3	1/100	4	.50	9
1/10%	2	1%	3	50	9
$100	1	$ .40	3	50%	6
1/99%	1	$ .10	2	.05	3
$ .10	1	25%	2	1/100	2
		.25	1	35%	1
		.001	1	2%	1
		400	1	1/1000	1
		1.90	1	100%	1
		1/4%	1	.002	1
		40%	1	1.40	1
		2%	1	2.40	1
		.02%	1	5	1
		$ .50	1	40	1
				.005%	1
				.055	1

An analysis of the wrong answers given reveals in many cases the confusion existing in the minds of the pupils. For example, the 34 pupils who gave "1/100%" as the answer for problem 1 obviously thought 1/100% was the same as 1/100 and those that gave ".01" must have thought that .01 was the same as .01%. They seem not to understand the fundamental character of percentage.

The proposed study will include not only a study of percentage as above, but also a study of fractions, decimals,

measurements, ratio, and the fundamental processes. An analysis of the work of pupils will be undertaken as well as a measurement of their accomplishment as illustrated in the study of Superintendent Kircher.

MATHEMATICS AND THE PUBLIC

By H. E. Slaught

It is safe to say that to the average man in the street mathematics means, for the most part, only a certain amount of arithmetic used in keeping his accounts, and a vague recollection of some things in algebra, geometry and possibly calculus which he may have studied in his school days. The public as a whole is quite oblivious to any great service which mathematical science has rendered to present-day civilization. There is nothing spectacular in the behavior of mathematics as is often the case with a physical or biological science. A new chemical combination producing a powerful explosive like TNT or a discovery in the biological field like insulin will be heralded the world around, while the most profound researches in mathematics which may underlie far-reaching developments in the practical affairs of life will be entirely unnoticed or will be accepted in the same commonplace way in which life-giving air is breathed.

However, in a span of twenty-five years it is possible to detect gradual changes in public appreciation of common blessings, and in the period 1900-1925 a great many things have happened with respect to mathematics in this country which indicate a steadily growing appreciation of its worth and power on the part both of its devotees and of the public in general.

In the secondary field the teachers themselves have experienced a remarkable awakening which began about 1900 with the organization of strong and active associations in New England, in the Middle Eastern states, and in the Central Western states, and which has continued unabated throughout the quarter century. This widespread movement reflects the

abiding confidence of the teachers in the worth and importance of their subject as well as their frank and honest recognition of its short-comings and their consequent determination to overcome whatever faults lay at their own door.

This merciless self-inspection culminated in the work of the National Committee on Mathematical Requirements which commanded the cooperation of thousands of teachers in all parts of the country banded together in more than one hundred mathematical organizations. Never was an investigation in this field better organized, more adequately financed and more painstaking in the effort to impartially weigh all phases of the questions involved. Probably no report of this kind ever received more open-minded consideration of its various recommendations by so large a proportion of the constituency. The immediate result of this close cooperation on the part of the great body of teachers of secondary mathematics naturally led to the federation of the various associations in the National Council of Teachers of Mathematics with the *Mathematics Teacher* as its official organ, and to this body is naturally committed the perpetuation of the work so ably inaugurated by the National Committee.

In the collegiate field the most notable event of the quarter century was the organization in 1916 of the Mathematical Association of America, a body whose membership now numbers over eighteen hundred. While this Association, whose official organ is the *American Mathematical Monthly,* is primarily concerned with the development and teaching of collegiate mathematics and with the stimulation of the beginnings of research, nevertheless it also recognizes the necessity of close cooperation with all efforts for improvement in the secondary field. Hence it was this Association which sponsored the National Committee and appointed the original nucleus of its membership with power to enlarge its number by inviting representatives from the various secondary organizations.

But this Association is now engaged in a still more direct effort to share with the public the fruits of mathematical dis-

coveries. Through funds generously provided by Mrs. Mary Hegeler Carus, the Association is preparing and publishing a series of monographs in which the expositions of various mathematical subjects are to be set forth in a manner comprehensive to a wide circle of thoughtful readers who, having a moderate acquaintance with elementary mathematics, wish to extend their knowledge without prolonged and critical study of the technical journals and treatises. The first of these monographs on the Calculus of Variations has already been published and the second on Functions of a Complex Variable is now in press. The third on Statistics is in preparation and well under way. It is expected that this series of monographs will continue indefinitely and it is ardently hoped that they will be the means of interpreting more and more of the wonders of mathematical science to a constantly enlarging body of the general public.

In the university field the outstanding feature of the quarter century just closed is the development of an American School of mathematical research fostered largely through the American Mathematical Society and contributed to chiefly by a few outstanding universities the number of which is gradually increasing. The output of American mathematical research is conveyed to the scientific public through the medium of four research journals, namely, the *Bulletin* and and the *Transactions* of the American Mathematical Society, the *American Journal of Mathematics* and the *Annals of Mathematics*. Research mathematical activities are further cared for through a division of the National Research Council and the National Academy of Sciences. This quarter century has thus seen America taking its place among the leading nations of the world in mathematical research.

In the foregoing, it must be admitted that the "public" referred to is very largely the mathematical public. The question still remains as to whether mathematics has ever "gotten across" to any appreciable extent to the public in

general. In what follows a few instances will be given which indicate some progress in this direction.

(1) When it was decided that we must enter the World War, it was natural that mathematicians, along with other loyal citizens, should volunteer to assist in their special capacity, but they were told by the Secretary of War that there seemed to be nothing needed in their line. It was not long, however, before the Government found that it *did* need the mathematicians, in fact, that it could not "carry on" without them in so important a matter as the effective development of the ordnance department. In a short time a corps of workers was organized under the leadership of a number of outstanding mathematicians and they speedily applied modern and powerful mathematical methods, both theoretical and practical, to those problems of ballistics which needed to be solved in order to properly equip our army. So effective had this service become at the time of the armistice, that the American forces were undoubtedly supplied with the best data of any of the armies for determining the effectiveness of gun fire. Other groups of mathematicians rendered similar effective service in developing submarine detection appliances. These two achievements were of vital importance in determining the outcome of the war and they revealed the power of mathematics in a most emphatic manner to the unsuspecting public.

(2) It came as a revelation to thousands of young men, many of whom had deliberately side-stepped all mathematical courses which they could possibly avoid in their school days, to find that those very courses were prerequisite to appointment or advancement as officers in either the army or the navy. It was a common experience to hear such men begging for the opportunity to enter classes in mathematics which they had previously ignored—and bragged about it too. The war served to elevate mathematics to a position of prominence not previously recognized by the casual public. This fact undoubtedly helped to swell the courses in mathematics in the colleges in the immediately succeeding years.

(3) But we are more interested in the arts and sciences of peace than in those of war. The example of the late Charles P. Steinmetz, whose name was a household word wherever the mysteries of electricity were under consideration, was probably more effective than any other means in driving home to the man in the street the fact that mathematics underlies all present day mechanical and physical progress. Thousands of boys and young men to whom the name of Steinmetz was magic have found early in their efforts to become electricians that one could climb a pole and splice a wire without knowing any mathematics, but that any masterful knowledge of electrical science could be gained only by the road that Steinmetz traveled, namely, by hard study of higher and higher mathematics. It is well known that his development of the mathematics of the alternating current, involving as it did those phases of higher mathematics which no one could have surmised in advance would be involved, did more than all other causes combined to lay the foundation upon which the great structure of the General Electric Company was built.

(4) It is well understood that the Rockefeller Foundation is now devoting its efforts chiefly toward the promotion of public health throughout the world. To this end it has not only made large grants to medical schools but it has indirectly stimulated independent work in the medical field by providing liberal research fellowships in the underlying sciences, physics and chemistry,—in chemistry as fundamental to medical research, and in physics as fundamental to chemistry. But it is not so widely known that the Foundation has recently extended the range of these fellowships to include mathematics as fundamental to both physics and chemistry. This step was taken deliberately after most careful consideration of the claims of mathematics as the basic science in this field.

This claim would hardly be questioned by the physical scientists themselves and would readily be admitted by the public. For example, it is said of the late Professor Chandler, dean of the Columbia School of Mines, when in the early days he wanted more mathematics put into the curriculum for

chemists and was opposed by the head of the department of mathematics, that he came into the faculty meeting with an armful of books and said: "These gentlemen, are important articles on chemistry which I cannot read because I do not know mathematics enough. We do not want our students to be in my predicament."

But the public would be staggered if such a claim were made for any of the biological sciences. And yet the writer knows a physician who is a diagnostic and research specialist of high standing, who besides his medical degree holds a Ph.D. in physiology from Harvard University, and who is now pursuing courses in advanced mathematics because he is unable to continue his researches successfully without more power in this line.

A few years ago the Mathematical Association of America invited several specialists in other sciences to take part in a symposium on "Mathematics in Relation to the Allied Sciences," and we were no little surprised to learn from biologists and biometoricians the wide range of mathematical subjects used by them in their research work. For instance, Dr. H. B. Williams of the College of Physicians and Surgeons, said in substance: "The biological sciences in the main rest entirely upon the fundamental sciences of physics and chemistry with all the mathematical foundation which these sciences presuppose." He cited his own investigations on "nerve impulse" and those by Einthoven of Leyden on "heart action" which were possible only because of their being well trained in methods of mathematical investigation.

One could proceed indefinitely citing evidence of a consciousness on the part of the public gradually awakening to the fact that mathematics is a marvelous and powerful instrument in the hands of trained thinkers for accomplishing great results. One further illustration, more or less amusing, was observed in the attempts of the newspaper reporters and popular magazine writers to describe "in a few words" the Einstein theory of relativity, but who invariably took refuge in the excuse that it all rested on mathematics of a "higher" type

than the readers could be expected to understand, and thus they were duly overcome with awesome respect both for relativity and for the underlying mathematics.

Finally, one other serious illustration of far-reaching significance should be mentioned. The great industries and large business corporations are coming to realize the importance of fostering research in connection with their special problems, and very many of them have established laboratories as regular departments of their activities. In numerous cases such organizations have found that mathematical research, usually of quite an elementary character but often far more intricate and technical than could have been predicted, has led to higher degrees of efficiency, to new phases of economy, and sometimes to complete reorganization of procedure. Many such corporations have become institutional members of the American Mathematical Society during the past two years, with the privilege of nominating to individual membership those representatives of their organizations who are interested and active in mathematics. This movement is bound to grow and to exert a continually widening influence in making mathematics known to and appreciated by the public.

One of the most significant evidences that the importance of mathematics is permeating the whole fabric of modern life is shown in the recent unparalleled development of the use of statistical methods in the study of quantitative relations in almost every department of investigation. This appears in the simplest form in all the proposed new curricula for the junior high schools. It is emphasized in the reorganized programs for the senior high schools. It is further developed in the enriched courses for college students not only in all the sciences but also in economics, sociology, anthropology, etc., and in most of the effective studies in education. It is indispensable in all laboratories everywhere, whether in the schools or in the factories, in commerce, in big business of every kind. In all such studies, if carried out to their fullest extent, it appears that the mathematics involved, while initially of the simplest nature, eventually becomes more and

more technical and requires the most careful reasoning in order to insure correct and dependable conclusions. These are phases of mathematics which are gradually growing into the consciousness of the general public and which are capable of recognition and widespread understanding.

RECREATIONAL VALUES ACHIEVED THROUGH MATHEMATICS CLUBS IN SECONDARY SCHOOLS

By Marie Gugle and Others

I. *Introduction.* The last twenty-five years have caused a new emphasis to be placed on recreational values. The rapidly changing conditions in industry and in economic life have forced us to give serious consideration to the worthy use of leisure. In Cardinal Principals of Secondary Education the proper use of leisure time is listed as one of the seven objectives. How can mathematics do its part in achieving desirable recreational outcomes? In passing, we may note that a good beginning has been made by a brilliant writer for the supplements of some of the Sunday papers. He is making a fortune syndicating our old friends, the puzzle problems of algebra. Recently he occupied a full page in scores of papers with simultaneous linear equations in three unknowns. Can we doubt any longer that there is human interest in these problems? But in this discussion we ask what are our mathematics teachers doing to realize recreational values?

Ever since pedagogy has modified belief in formal discipline and drudgery by an emphasis upon interest and appreciation, teachers of mathematics in secondary schools have tried to present the subject to pupils in more attractive forms. In the class room real problems have been urged. The dramatic element has been brought in wherever possible, especially in lower grades; e. g., in having grocery stores and banks. The kindergarten principle of learning through play has become more universal by extending the recreational phase of mathematics.

The use of mathematical recreations as a devise is not a new idea. Puzzles and catch questions were in the old arith-

metics. The human mind has always found pleasure in puzzles, tricks, and curiosities of all sorts. This tendency is in every person, young and old, of every race and of every time.

Cantor attributes the first mathematical puzzle to Ahmes, 2000 B.C. The problem of the fox, the goose, and the sack of corn was known to Alcuin in the time of Charlemagne. The hare and hound problem appears in Italian arithmetics of 1460. Magic squares were known to the Arabs and Hindus. There is a record of one in a Chinese book of the date of 1125.

Ball's Mathematical Recreations, White's Scrap Book, and Jones' Mathematical Wrinkles are based on German, French, and English publications of 1507, 1612, and 1694. We see, therefore, that mathematical recreations are almost as old as mathematics itself. They have always been a source of pleasure and profit to both pupil and master. In the light of history we cannot regard these phases of mathematics as too trivial for pupils of today.

II. *Development in Secondary Schools.* One of the first mathematical clubs in secondary schools was organized nearly twenty-five years ago in the Shattuck School, a private school for boys at Fairbault, Minnesota. In an article in School Science and Mathematics, Mr. C. W. Newhall describes the organization. Fifteen boys from the senior class were the instigators. The club held evening meetings every two weeks.

Before 1912, the mathematics classes of Horace Mann School were organized into clubs, whose meetings were held during certain recitation periods.

In 1913 Miss Marie Gugle, then a teacher at Scott High School, Toledo, Ohio, organized a Euclidean Club among the boys of grades ten to twelve, whose ratings in mathematics were excellent or good. This club still exists as an active organization. Its meetings were held in the evenings. Its programs usually had three features: a biographical sketch of some great mathematician and a story of his contributions; a mathematical game, trick, fallacy, or unique solution; and an

account of some scientific discovery or invention related to mathematics.

In 1914, groups of interested pupils in Hyde Park, Wendel Phillip's and Bowen High Schools of Chicago began giving time after school to informal discussions and to the investigation of suggested problems.

By 1916, clubs were rather widely scattered. In Marion, Ohio, one was organized to answer the question, "What shall be done for the bright pupil?" Miss Irene Brown tells of a club in a Girls' School in England, in order that for them the bypaths of mathematics might be illuminated. In this same year, the first club in Columbus, Ohio, was organized in Roosevelt Junior High School. It took the same name and pin as had been adopted by the club in Toledo. Very soon this club had to have three chapters, Alpha, Beta, and Gamma chapters for the ninth, eighth, and seventh grades respectively. Practically all of the twelve junior high schools in Columbus have one or more mathematics clubs. Three of the five senior high schools have such organizations.

III. *Objectives and Results.* The purposes of a mathematics club are "to illuminate the bypaths of mathematics"; to study certain interesting matters connected with mathematics which do not find a place in the usual class room; to promote interest in the study of mathematics; to give the pupils glimpses of the future and incentives to further study; to develop an appreciation for the truth and beauty in mathematics and our dependence upon it in practical life; and to furnish an outlet for pupils' social instincts.

It is the consensus of opinion of those teachers who have directed mathematics clubs that the foregoing objectives are readily realized and that the transfer of interest and initiative that carries over into the classroom more than repays for their time and effort.

IV. *Organization and Membership.* There are various types of organization. Some clubs include the entire class or several combined in their membership. Others are limited.

either to those whose ratings in scholarship are superior or to the pupils' classification or grade in school, as eighth grade or tenth grade club.

Some clubs meet during school hours, either in a club activity period or in the regular recitation hour. Others meet after school or in the evenings. The frequency of meetings varies from once a week to once a month.

The names given to clubs are of different types:

A. Greek letters; as Mu Alpha Theta

B. Mathematicians; as, Pythagorean, Euclidean, Archimedian Clubs. Some have been named for teachers in their schools.

C. Mathematical Figures; as, Magic Circle, Triangle, Octagon, Hypocycloid Clubs.

Usually mathematics clubs adopt their emblems, as in pins, banners, and colors. For pins the following devices have been used; the pentagram, the old mathematical instrument known as a quadrant, a combination in a triangle of an overlapping semicircular protractor, compass, and pentagram, and a circle with a hypocycloid.

V. *Program Material.* Once a teacher of Latin who had a thriving Latin Club, asked how one could find enough material to keep up interest in a mathematics club. She little knew that the supply is inexhaustible and that the only difficulties are those of selection and adjustment to suit the ages and educational advancement of the members.

Some of the same topics may be used for seventh or ninth grade clubs, but the assignments must be different. For example, in a seventh grade club a program on magic squares should include only those with three, five, or seven on a side. Papers should be provided and members should make their own under the direction of the speaker. The same topic on a ninth grade program should include a sketch of their history and the making of magic squares with both odd and even number of squares on a side, and magic circles.

The following programs adapted to junior high school grades may be suggestive:

A. For Seventh Grade Club

1. (a) Nine, a Magic Number
   White: Scrap Book, page 25.
   (b) What a Billion Means
   White: Scrap Book, pages 9 and 10.
   (c) Who was Thales?
2. (a) Archimedes, the Mathematician, and some of his Inventions
   (b) Mysterious Addition
   Jones: Mathematical Wrinkles, page 103.
   (c) Number Tricks
   Material supplied from Popular Science Monthly or other magazines by pupils.
3. (a) Arithmetic Tricks
   (b) Multiplication
   Select one digit out of 12345679 and multiply by 9 times the digit. Result is row of same digits. See Jones' Mathematical Wrinkles, page 79.
   (c) Paradox Party
   Dudeney, Amusements in Mathematics, page 137.

B. For Eighth Grade Club

1. **Hallowe'en Program**
   (a) Club Songs
   (b) Apparition of Two Ghosts, Descartes and Pythagoras
   The ghosts meet and exchange stories of what each did while on earth.
   (c) Clever Question Contest
   Leaders choose sides. Questions for contest are selected from Jones' Mathematical Wrinkles.
2. (a) Reading: Number Stories of Long Ago
   Read parts of Chapter One. If too long, certain parts may be told in abridged form.
   (b) Remarkable Numbers
   Teachers College Record, November, 1912, or Smith, Number Stories of Long Ago, pages 105-107.
   (c) A Number Trick
   Selected from Mathematical Wrinkles or Ball's Mathematical Recreations.
3. (a) How our Hindu-Arabic Numerals Grew
   Smith: Number Stories of Long Ago, pages 13-43. (Make charts of illustrations.)
   (b) Where did the signs, $+$, $-$, $\times$, $\div$, and $=$ come from?
   Ball: History of Mathematics.

C. For Ninth Grade Club

1. (a) Picture of World without any Mathematics
   (b) The Part Mathematics Plays in Our Everyday Lives

(c) Fallacy: Prove that you are as old as Methuselah
Jones: Mathematical Wrinkles.

2. (a) Fallacy: Prove that 1 equals 2
(b) How the Algebraic Symbols Grew
(c) Tangrams
Dudeney: Amusements in Mathematics, page 43. Encourage originals.

3. (a) Brief Discussion about Napier and his Rods
(b) Short Talk on the Slide Rule
(c) Divide the club into three groups, each with a leader who is expert in the game for his group. After his instruction, the members of the group enter a contest to see who can play the game most skillfully. The three games are with Napier's Rods, the slide rule, and the circular slide rule. Sometimes very simple prizes are offered in the various contests.

For additional topics for program material, see the following references:

Minerva Guldon et al., Mathematics Club Programs, Mathematics Teacher, October 1924.

Marie Gugle, Modern Junior Mathematics, Book Two, Appendix, Gregg Publishing Company.

Zulu Reed, High School Mathematics Clubs, Mathematics Teacher, October 1925.

VI. *Minimum Reference Library for Clubs.* For a newly organized club, the following list of books would form a satisfactory minimum reference library for junior and senior high schools. It might be a nucleus for a more extensive one to be built up gradually.

1. Ball: Primer of the History of Mathematics, Macmillan Co. 60c.
2. Jones: Mathematical Wrinkles, Revised, Life and Casualty Bldg., Nashville, Tenn. $2.10.
3. Smith: Number Stories of Long Ago. Ginn and Co.
4. White: Scrapbook of Elementary Mathematics. Open Court Publishing Co. $1.00.
5. Portraits of Mathematicians. Open Court Publishing Co.

VII. *Plays for Mathematics Clubs.* In the Mathematics Teacher of October, 1924, Miss Alma E. Crawford has a very attractive little play entitled "A Little Journey to the Land of Mathematics."

A number of Columbus teachers have found that, with some suggestions, groups of pupils are capable of writing very interesting and instructive plays and pageants.

VIII. *Bibliographies.* Additional reference books and magazine articles may be selected from the following bibliographies. These references cover the organization of clubs, history of mathematics, and mathematical recreations including games, tricks, and plays.

1. Gugle, Modern Junior Mathematics, Book Two.
2. Gugle, et al., Mathematics Club Programs, Mathematics Teacher, October 1924.
3. National Committee Report, pages 625-627. List of magazine articles only.
4. Reed, High School Mathematics Club, Mathematics Teacher, October 1925. List of books and magazines.
5. Smith, et al., List of mathematics books, annotated, Teachers College Record, April 1925. List of books only.

Teachers who are directing clubs in secondary schools will find, in addition to the foregoing books, a continuous supply of program material in the two magazines for mathematics teachers:

1. *The Mathematics Teacher*, the official publication of the National Council of Teachers of Mathematics. Membership in the Council, including the subscription to the magazine, is two dollars ($2.00) a year. Editor, J. R. Clark, Teachers College, New York.
2. *School Science and Mathematics*, the official organ of the Central Association of Science and Mathematics Teachers. Membership and subscription to the magazine is two and a half dollars a year ($2.50). Editors, Smith and Turton, 2055 East 72nd Place, Chicago, Illinois.

Valeria Bostwick, Eleventh Avenue Jr. High School
C. B. Marquand, Indianola Jr. High School
Helen Marquand, Vice Principal of Mound St. Jr. High School
Amy F. Preston, Roosevelt Jr. High School
Marie Gugle, Chairman, Assistant Superintendent of Schools, Columbus, Ohio.

MATHEMATICS BOOKS PUBLISHED IN RECENT YEARS FOR OUR SCHOOLS AND FOR OUR TEACHERS

By Edwin W. Schreiber

Note: The compiler of this bibliography thought it well that the publishers should have first say in the matter of presenting their books. He therefore sent a letter to forty publishing houses asking them for certain information about mathematics books published since 1920, for secondary schools. Many of the firms responded very graciously and to them gratitude is hereby expressed. Others failed to reply and so if there are valuable omissions we are sorry. Some of the noticeable gaps were filled in by the compiler through the excellent facilities afforded by the John Crerar Library of Chicago. It was our aim to list only those books which have been published since 1920. An attempt has been made to answer the following questions concerning each publication: (1). What is the exact title of the book? (2) Who is the author and where does he teach? (3) How many pages in the book and how large is it? (4) When was it published? (5) What is the price? (6) Who publishes the book?

JUNIOR HIGH SCHOOL MATHEMATICS

1. Bonser, Pickell, Smith. *Practical Mathematics for Junior High Schools*, by Frederick G. Bonser, Prof. of Ed., Columbia U., Frank G. Pickell, Supt. of Schools, Montelair, N. J., and James H. Smith, Supt. of Schools, West Aurora, Ill., Book I, 246p, 1924, 88c.
2. Book II, 278p, 1924, 88c.
3. Book III, 294p, 1924, 88c.
4. Breslich. *Junior Mathematics*, by Ernest R. Breslich, Head of Math. Dept., University High School, Chicago. Book I, 264p, 1925, 96c. Macmillan Co.

5. Drushell, Withers. *Junior High School Mathematical Essentials*, by J. Andrew Drushel, Dept. of Math., New York Univ., and John W. Withers, Dean of the School of Ed., New York Univ. 7th Yr., 196p, 1924, 88c.
6. 8th Yr., 237p, 1924, 96c. Lyons and Carnahan.
7. Gonnelly, Huff. *7th Year Mathematics*, by Joseph F. Gonnelly, Dist. Supt. in charge of Junior High Schools, Chicago, and L. Grace Huff, Inst. of Math., Lane Technical High School, Chicago. 239p, 1925, 96c. Charles E. Merrill Co.
8. Gugle. *Modern Junior Mathematics* (New Ed.), by Marie Gugle, Ass't Supt. of Schools, Columbus, Ohio. Book I, 310p, 1924, 80c.
9. Book II, 239p, '1924, 90c.
10. Book III, 298p, 1924, $1.10. The Gregg Publishing Co.
11. Hart. *Junior High School Mathematics*, by Walter W. Hart, Ass't Prof of Math., Univ. of Wis. Book I, 226p, 1921, 80c.
12. Book II, 256p, 1922, 88c.
13. Book III, 346p, 1923, $1.28. Houghton Mifflin Co.
14. Schorling, Clark. *Modern Mathematics*, by Raleigh Schorling, Head of Math. Dept., Univ. High School, Univ. of Mich., and John R. Clark, Lincoln School of Teachers College, Columbia Univ. 7th Yr., 256p, 1924, 88c.
15. 8th Yr., 254p, 1924, 88c.
16. *Modern Algebra*, 9th Yr., 382p, 1924, $1.36. World Book Co.
17. Schorling, Clark, Lindell. *Instructional Tests in Algebra*, by Raleigh Schorling, John R. Clark, and Selma Lindell, The Univ. High School, Ann Arbor, Mich., 72p, 1925, 30c. George Wahr, Publisher, Ann Arbor, Mich.
18. Thorndike. *Thorndike Junior High School Mathematics*, by E. L. Thorndike, Prof. of Ed., Teachers College, Columbia Univ. Book I, 196p, 1925, 80c.
19. Book II, 196p, 1925, 80c. Rand, McNally and Co.
20. Vosburgh, Gentleman, Hassler. *Junior High School Mathematics, Revised*, by Wm. L. Vosburgh, Mead of Math. Dept., Boston Normal School, the late Frederick W. Gentleman, Dept. of Math., Mech. Arts High School, Boston, and J. O. Hassler, Prof. of Math., Univ. of Oklahoma, 1st Course, 228p, 1924, 88c.
21. 2nd Course, 254p, 1924, $1.00. Macmillan Co.
22. Wentworth, Smith, Brown. *Junior High School Mathematics, Rev. Ed.*, by George Wentworth, late author of a series of Math. Texts, David Eugene Smith, Prof. of Math., Teachers College, Columbia Univ., and J. C. Brown, Pres. State Teachers College, St. Cloud, Minn. Book I, 266p, 1925, 92c.
23. Book II, 282p, 1925, $1.00. Ginn and Co.

ARITHMETIC

24. Brown, Eldredge. *The Brown-Eldredge Arithmetics*, by J. C. Brown, Pres. Teachers' College, St. Cloud, Minn., and A. C. Eldredge, Ass't Supt. of Schools, Cleveland, Ohio. Book III, 302p, 1925, 88c. Row, Peterson and Co.

25. Chadsey, Smith. *Efficiency Advanced Arithmetic*, by Charles E. Chadsey, formerly Supt. of Schools, Detroit, Mich., Dean, Col. of Ed., U. of Ill., and James H. Smith, Prof. of Math., State Normal School, Whitewater, Wis. 346p, 1920, 92c. Mentzer, Bush and Co.

26. Gannon, Jackson. *Drill Exercises and Problems in Arithmetic*, by John F. Gannon, Supt. of Schools, Pittsfield, Mass., and Joseph Jackson, former Prin. High School of Commerce, Worcester, Mass. Book II, 196p, 1923, 68c. D. C. Heath and Co.

27. Hoyt, Peet. *Everyday Arithmetic*, Advanced book by Franklin S. Hoyt, formerly Ass't Supt. of Schools, Indianapolis, Ind., and Harriet E. Peet, Inst. in Methods of Teaching Arith., State Normal School, Salem, Mass. 326p, 1920, 88c. Houghton Mifflin Company.

28. Miner, Elwell, Touton. *Business Arithmetic*, by George W. Miner, formerly head of Com. Dept., High School, Westfield, Mass., Fayette H. Elwell, C. P. A., Prof. of Accounting, Univ. of Wis., and Frank C. Touton, Prof. of Ed., Univ. of Southern California. 410p, 1923, $1.48. Ginn and Co.

29. Miner, Elwell, Touton. *Essentials of Business Arithmetic*, 236p, 1924, $1.20. Ginn and Co.

30. Neely, Killins. *Modern Applied Arithmetic*, by R. R. Neely, Supervisor of Part Time Schools, Peoria, Ill., and James Killins, Director of Vocational Ed., Johnstown, Pa. 156p, 1921, $1.00. Blakiston.

31. Schorling, Clark. *Practice Exercises for Accuracy and Speed Timed Instructional Tests for Seventh and Eighth Grades.* 64p, 1922, 15c. Gazette Press, Yonkers, New York.

32. Stone. *Stone's Advanced Arithmetic*, by John C. Stone, Head of Math. Dept., State Normal School, Montclair, N. J. 326p, 1925, 80c. B. H. Sanborn and Co.

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